Sets

Monday, August 22, 2022 1:59 PM

Let S be a set

Axiom of Extension: A set is completely defined by what the elements are

$\{1,2,3\} = \{3,2,1\} = \{1,3,2,3,1\}$

$\{1,\{1\}\} \neq$

$\mathbb{Z}$ set of all integers

$\mathbb{Q}$ set of all rational

$\mathbb{R}$ set of all real

$\mathbb{R}^+$ set of all positive real

$\mathbb{R}^-$ set of all negative real

$\mathbb{R}^{nonneg}$ set of all nonnegative real

$\{x \in S \mid P\}$ the set of all x in S such that P is satisfied

$\{x \in \mathbb{Z} \mid 0 \le x < 27\}$ the set of all nonnegative integers less than 27

$A \subseteq B$ A is a subset of B

$A \not\subseteq B$ A is not a subset of B

$(A \subseteq B)$ & $(B \not\subseteq A)$ Proper subset

Ordered Pair

$(a,b) = (c,d)$ if $a=c$ and $b=d$

$(a,b) = \{\{a\},\{a,b\}\}$

Ordered n-tuples

$(x_1, x_2, \ldots x_n)$

Cartesian Product

$A_1 \times A_2 \times \ldots \times A_n$

A = {apple, banana, lemon}

B = {aardvark, bear, camel, deer}

A×B = {(apple, aardvark)
(apple, bear)
(apple, camel)
(apple, deer)
⋮
(lemon, ardvark)
(lemon, bear)
(lemon, camel)
(lemon, deer)}

(lemon, camel)
(lemon, deer) }

A = {0,1} ⇒ bit string

A = {p, q, r} such that length three containing two or more p's

p,p,p
p,p,r
p,p,q
r,p,p
q,p,p
p,r,p
p,q,p

Relations

x R y = (x,y) ∈ R

x R̸ y = (x, y) ∉ R

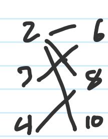
A is domain of R and B is the codomain of R

A = {2, 3, 4} and B = {6, 8, 10}

(x, y) ∈ R means that $\frac{y}{x}$ is an integer

R : {(2, 6) (2,8), (2,10), (3,6), (4,8)}

Domain : A
Codomain : B



Functions

1. every element of A is the first element in ordered pair in F
2. no two distinct ordered pairs have the same first element

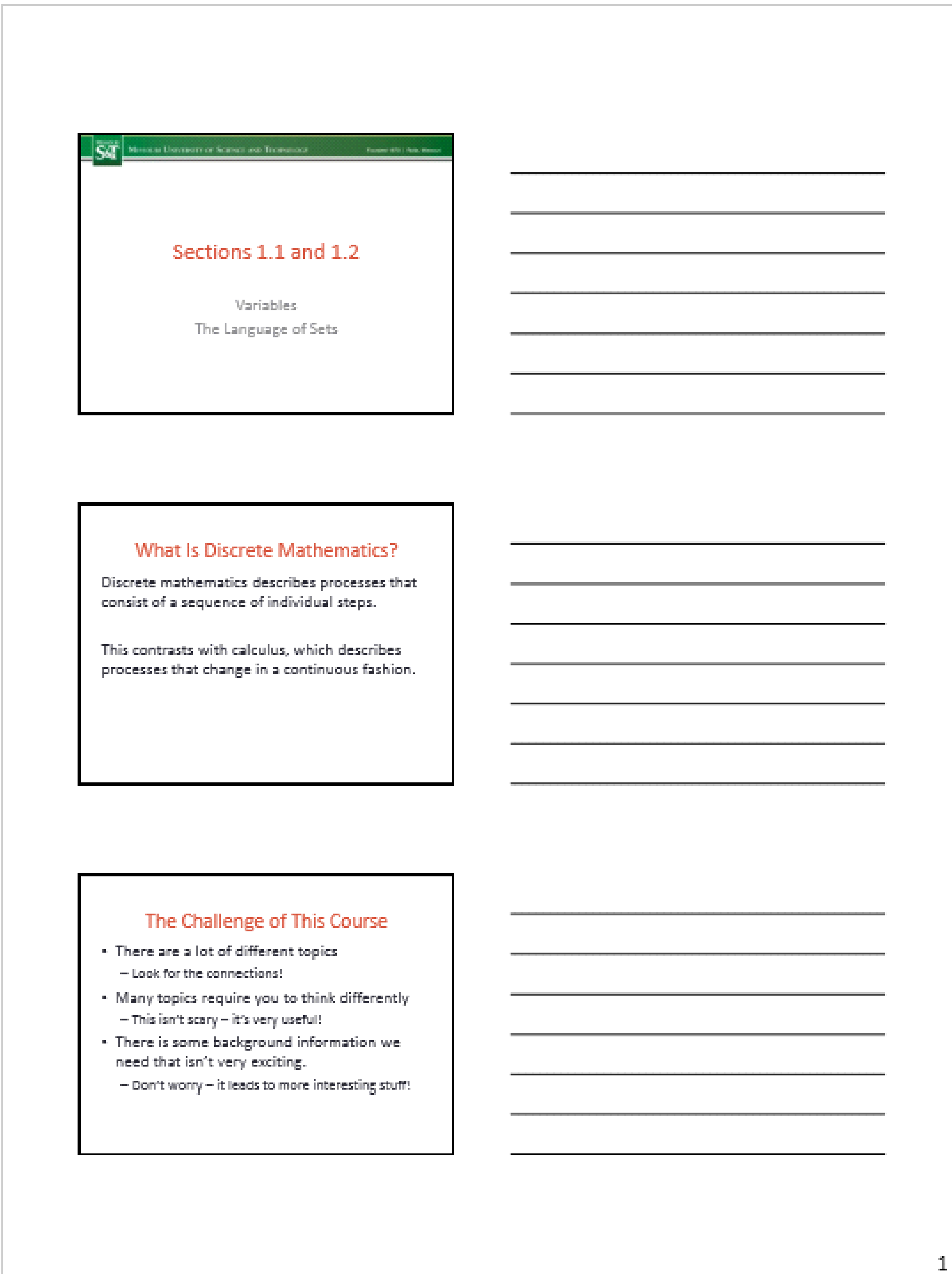
Graphs : Two finite sets, V(G) vertices and E(G) edges,

Directed graph

Variables and Sets

Friday, August 26, 2022 1:56 PM

CS1200+Lecture+1+H...



Sections 1.1 and 1.2

Variables
The Language of Sets

What Is Discrete Mathematics?

Discrete mathematics describes processes that consist of a sequence of individual steps.

This contrasts with calculus, which describes processes that change in a continuous fashion.

The Challenge of This Course

- There are a lot of different topics
  - Look for the connections!
- Many topics require you to think differently
  - This isn't scary – it's very useful!
- There is some background information we need that isn't very exciting.
  - Don't worry – it leads to more interesting stuff!

1

My Prediction

For many of you, this will be the most interesting – and enjoyable – mathematics course you've taken!

Variables

A variable is a placeholder to use when you want to talk about something but either

- You imagine that it has one or more values but don't know precisely what they are
  - Typical idea in algebra or programming

or

- You want whatever you say about it to be equally true for all elements in a given set while avoiding ambiguity
  - Powerful idea in discrete math

Example 1

Use variables to rewrite the sentence more formally and remove ambiguity.

If the cube of a real number is nonnegative, then it is nonnegative.

2

Important Types of Mathematical Statements

A universal statement says that a certain property is true for all elements in a set.

Examples:

- All oranges are fruits.
- Every even number is divisible by 2.

Important Types of Mathematical Statements

A conditional statement says that if one thing is true then some other thing also must be true.

Examples:

- If your lunch contains an orange, then it contains a fruit.
- If a number is divisible by 12, then the number is divisible by 4.

Important Types of Mathematical Statements

An existential statement says that a property holds for at least one element of a set.

Examples:

- There is a caffeinated beverage in the Coke machine.
- There is a number between 20 and 30 which is divisible by 12.

Combining Statement Types

Universal Conditional Statements are both universal (applicable to every element in a set) and conditional (has an if-then component).

Example:

- For every athlete *T*, if *T* is a quarterback, then *T* is a football player.

Example 2

Rewrite the universal conditional statement in a way which makes its conditional nature explicit but its universal nature implicit.

For every athlete *T*, if *T* is a quarterback, then *T* is a football player.

Example 3

Rewrite the universal conditional statement in a way which makes its universal nature explicit but its conditional nature implicit.

For every athlete *T*, if *T* is a quarterback, then *T* is a football player.

Combining Statement Types

Universal Existential Statements are first universal (applicable to every element in a set) and then existential (asserting the existence of something).

Example:

- Every football team has a tallest player.

Combining Statement Types

Existential universal statements first assert that an object exists and then indicate the object satisfies a certain property for all things of a certain kind.

Example:

- There is a player who is at least as tall as every person on the football team.

The Language of Sets

Let S be a set.

- $x \in S$ means x is an element of S
- $x \notin S$ means x is not an element of S

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Set-Roster Notation

Set-roster notation involves writing all elements of a set between set brackets.

Examples:

$\{1,2,3,4,5\}$

$\{1,2,3, \ldots, 50\}$

$\{1,2,3, \ldots\}$

The Axiom of Extension

A set is completely defined by what its elements are.

Important Implications:

- The order in which elements are listed does not matter.
- Elements may be listed more than once without impacting the nature of the set.

Example 4

What is the difference between the following sets?

$\{1,2,3,4,5\}$

$\{5,4,3,2,1\}$

$\{1,3,5,2,3,4\}$

nothing

6

Example 5

How many elements are in each set?

{1,1,2,2,3} 3

{1, {1}} 2

Special Sets

$\mathbb{Z}$ The set of all integers

$\mathbb{Q}$ The set of all rational numbers
Recall: Rational numbers are quotients of two integers.

$\mathbb{R}$ The set of all real numbers

Special Sets

$\mathbb{R}^+$ The set of positive real numbers

$\mathbb{R}^-$ The set of negative real numbers

$\mathbb{R}^{nonneg}$ The set of nonnegative real numbers

Similar notation works with $\mathbb{Z}$ and $\mathbb{Q}$

7

Set-Builder Notation

Let S denote a set and let P be a property that elements of S may or may not satisfy. Then, we can define a set as

$$\{x \in S \mid P\}$$

which means
"the set of all x in S such that P is satisfied".

Example 6

Write the following set using set-builder notation.

The set of all nonnegative integers less than 27.

$\{x \in \mathbb{Z} \mid 0 \le x < 27\}$

Subsets

If A and B are sets, then A is called a subset of B (denoted $A \subseteq B$) if and only if every element of A is also an element of B.

Formal definition:
$A \subseteq B$ means that for every element x, if $x \in A$ then $x \in B$.

Subsets

For A to not be a subset of B (denoted $A \not\subseteq B$), at least one element of A is not an element of B.

Proper Subsets

If A and B are sets, then A is called a proper subset of B (denoted $A \subset B$) if and only if every element of A is also an element of B but there is at least one element of B which is not an element of A.

Example 7

Which of the following are true statements?

$2 \in \{1,2,3\}$ ✓	$2 \subseteq \{1,2,3\}$ ✗
$\{2\} \in \{1,2,3\}$ ✗	$\{2\} \subseteq \{1,2,3\}$ ✓
$2 \in \{\{1\},\{2\},\{3\}\}$ ✓	$2 \subseteq \{\{1\},\{2\},\{3\}\}$ ✗
$\{2\} \in \{\{1\},\{2\},\{3\}\}$ ✗	$\{2\} \subseteq \{\{1\},\{2\},\{3\}\}$ ✓

Sets, Relations, Functions, Graphs

Friday, August 26, 2022 1:56 PM

CS1200+Lecture+2+H...

Sections 1.2, 1.3, and 1.4

The Language of Sets,
Relations and Functions, and Graphs

Ordered Pairs

Given elements a and b, the symbol (a, b) denotes the ordered pair consisting of a and b together with the specification that a is the first element of the pair and b is the second element.

Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Ordered Pairs as Sets

The ordered pair (a, b) is a set of the form

$$\{\{a\}, \{a, b\}\}$$

If a and b are distinct, then the two sets are distinct and a is in both whereas b is only in the second set, allowing us to distinguish between a and b and imply an ordering.

1

Ordered n-tuples

The ordered n-tuple

$$(x_1, x_2, \dots, x_n)$$

consists of the (not necessarily distinct) elements $x_1, x_2, \dots, x_n$ together with the defined ordering.

Cartesian Products

Given sets $A_1, A_2, \dots, A_n$, the Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n$$

of $A_1, A_2, \dots, A_n$ is the set of all ordered n-tuples

$$(a_1, a_2, \dots, a_n)$$

where $a_1 \in A_1$, $a_2 \in A_2$, ..., $a_n \in A_n$.

Example 1

Let $A = \{\text{apple}, \text{banana}, \text{lemon}\}$
and $B = \{\text{aardvark}, \text{bear}, \text{camel}, \text{deer}\}$

Find $A \times B$

Strings

Let n be a positive integer. Given a finite set A, a string of length n over A is an ordered n-tuple of elements of A written without parentheses or commas.

The elements of A are called the characters of the string.

Strings

The null string over A is defined to be the string with no characters, sometimes denoted λ, and is said to have length zero.

If $A = \{0,1\}$, then a string over A is said to be a bit string.

Example 2

Let $A = \{p, q, r\}$. List all strings of length three over A which contain two or more p's.

Relations

A relation R from a set A to a set B is a subset of $A \times B$.

Given an ordered pair (x, y) in $A \times B$, we say x is related to y by R if and only if (x, y) is in R.

$x\ R\ y$ means that $(x, y) \in R$

$x \not R\ y$ means that $(x, y) \notin R$

A is the domain of R and B is the codomain of R.

Example 3

Let $A = \{2,3,4\}$ and $B = \{6,8,10\}$ and define a relation R from A to B as follows:

$(x, y) \in R$ means that $\frac{y}{x}$ is an integer.

a) Write R as a set of ordered pairs.
b) What are the domain and codomain of R?
c) Draw an arrow diagram for R.

Functions

A function F from a set A to a set B is a relation with domain A and codomain B which satisfies the following two properties.

1. For every element $x \in A$, there is an element $y \in B$ such that $(x, y) \in F$.
2. For all elements $x \in A$ and $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$, then $y = z$.

Functions (less formally)

A relation F from A to B is a function if and only if

1. Every element of A is the first element of an ordered pair in F

and

2. No two distinct ordered pairs in F have the same first element.

Example 3 (continued)

Let $A = \{2,3,4\}$ and $B = \{6,8,10\}$ and define a relation R from A to B as follows:

$(x, y) \in R$ means that $\frac{y}{x}$ is an integer.

d) Is R a function?

Example 4

Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows:

For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,
$(x, y) \in C$ means that $x^2 + y^2 = 1$.

Is C a function? Explain.

Function Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $F(x)$, which is read "F of x".

Example 5

Define functions f and g from $\mathbb{R}$ to $\mathbb{R}$ by the following formulas:

For every $x \in \mathbb{R}$,

$f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$

Does $f = g$? Explain.

Example 6

Abbie lives in Lemon Grove. Her friends there are Bob, Cindy, and Diana.

Abbie used to live in Mango. Her friends there were Esther, Fran, and George.

Diana and Fran are cousins. They used to live in Niceville where their best friend was Harriet. Harriet's brother is Ian, and Ian is his own best friend.

Cindy's roommate is Janice, who met Harriet at a summer camp in Orange Beach.

Draw a diagram to represent these relationships.

Graphs

A graph G consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges, where each edge is associated with a set consisting of either one or two vertices called its endpoints.

The correspondence from edges to endpoints is called the edge-endpoint function.

Graphs

An edge with just one endpoint is called a loop.

Two or more distinct edges with the same set of endpoints are said to be parallel.

An edge is said to connect its endpoints.

Two vertices connected by an edge are adjacent.

A vertex that is an endpoint of a loop is adjacent to itself.

Graphs

An edge is said to be incident on each of its endpoints.

Two edges incident on the same endpoint are called adjacent.

A vertex on which no edges are incident is called isolated.

The Degree of a Vertex

If G is a graph and v is a vertex of G, then the degree of v, denoted $\deg(v)$, equals the number of edges incident on v. An edge that is a loop is counted twice.

Directed Graphs

A directed graph, or digraph, consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $D(G)$ of directed edges, where each edge is associated with an ordered pair of vertices called its endpoints.

If edge e is associated with the pair (v, w) of vertices, e is the directed edge from v to w.

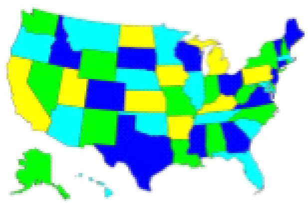
Example 7

Color Missouri and its 8 adjacent states using the fewest colors possible.

Example 7

Color Nevada and its 5 adjacent states using the fewest colors possible.

Example 7



Color the rest of the map using the fewest colors possible.

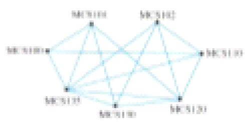
The Four-Color Theorem

Any geographic map, however complex, can be colored using just four colors such that no two adjacent regions have the same color.

Example 8

A department wants to schedule finals so that no student has more than one exam on a given day. The vertices of the graph on the next slide show the courses being taken by more than one student, with an edge connecting two vertices if there is a student in both courses. Find a way to color the vertices of the graph so that no two adjacent vertices have the same color and explain how to use the result to schedule the finals.

Example 8



Logical Form and Equivalence

Friday, August 26, 2022 1:57 PM

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Section 2.1

Logical Form and Logical Equivalence

What Is Logic?

Logic is the science of necessary inference.

Logical analysis will *not* help you determine the intrinsic merit of an argument's content.

Logical analysis *will* help you analyze an argument to determine whether the truth of the conclusion follows necessarily from the truth of the premises.

Statements

A statement is a sentence which is either true or false but not both.

1

Example 1

Which of the following sentences are statements?

a) Joe is a physics major ✓
b) $4^2 = 8$ ✓
c) $x^2 = 9$ ✗

Example 2

Represent the common form of the argument using letters to stand for component sentences.

If the program syntax is faulty, then the computer will generate an error message.

If the computer generates an error message, then the program will not run.

Therefore, if the program syntax is faulty, then the program will not run.

Symbols in Logical Expressions

∴ Therefore
∧ And
∨ Or
~ Not (sometimes ¬ is used instead)

But

Sometimes *but* is used in place of *and* when the second part of the sentence is somehow surprising.

Example 2

Consider the following sentence:
Steve has long arms but he is not tall.

Translate from English to symbols, letting
l = Steve has long arms
t = Steve is tall

$l \wedge \sim t$

Neither-Nor

Neither p nor q means $\sim p \wedge \sim q$.

Example 3

Let p represent $1 < x$.
Let q represent $x = 1$.
Let r represent $x < 7$.

Write the following inequality as a logical statement.

$1 \leq x < 7$

(p v q) ∧ r

Truth Values

For sentences to be statements, they must have well-defined truth values – they must be either true or false.

Our goal is to analyze the truth of compound statements based on the truth values of the statements which compose them.

Negation

If p is a statement variable, the negation of p is $\sim p$ (read "not p").

$\sim p$ has the opposite truth value from p.

p	$\sim p$
T	F
F	T

Conjunction

If p and q are statement variables, the conjunction of p and q is $p \wedge q$ (read "p and q").

$p \wedge q$ is true if and only if both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

If p and q are statement variables, the disjunction of p and q is $p \vee q$ (read "p or q").

$p \vee q$ is false if and only if both p and q are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statements

A statement form (or propositional form) is an expression made up of statement variables and logical connectives (such as $\wedge$, $\vee$, and $\sim$) that becomes a statement when actual statements are substituted for the component statement variables.

The truth table for a given statement form displays the truth values that correspond to all possible combinations of truth values for the component statement variables.

Example 4

Construct a truth table for the statement.

$p \vee (p \wedge q)$

p	q	p∧q	p∨(p∧q)
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Logical Equivalence

Two statement forms are logically equivalent if and only if they have identical truth values for every possible substitution of statements for their statement variables.

If P and Q are logically equivalent, we write $P \equiv Q$.

Example 4 (continued)

Determine whether the following statements are logically equivalent.

1. $p \vee (p \wedge q)$
2. p

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Example 5

Determine whether the following statements are logically equivalent.

1. $(p \vee q) \vee (p \wedge r)$
2. $(p \vee q) \wedge r$

NO

p	q	r	p∨q	p∧r	(p∨q)∨(p∧r)	(p∨q)∧r
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	F	T	T
F	T	F	T	F	T	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

De Morgan's Laws

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	~p	~q	p∧q	~(p∧q)	~p∨~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Example 6

Use De Morgan's Laws to write negations for each statement.

Jennifer is late to class or my watch is fast.

Hal is fat and Hal is happy.

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Tautologies

A tautology is a statement form that is always true regardless of the truth values of the individual statements.

Simple example:

$p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Contradictions

A contradiction is a statement form that is always false regardless of the truth values of the individual statements.

Simple example:

$p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Example 7

Write a truth table for the statement form. Is it a tautology, a contradiction, or neither?

$(p \wedge \sim q) \wedge (\sim p \vee q)$

p	q	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	T	F	F
T	F	T	F	F
F	T	F	T	F
F	F	T	F	F

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Conditional Statements

Monday, August 29, 2022 1:58 PM

CS1200+Lecture+4+H...

Section 2.2

Conditional Statements

Conditional Statements

If p and q are statement variables, the conditional of q by p is $p \rightarrow q$ (read "if p then q" or "p implies q").

$p \rightarrow q$ is false when p is true and q is false; otherwise, it is true.

p is called the hypothesis (or antecedent).

q is called the conclusion (or consequent).

Conditional Statements

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Vacuously True Statements

Consider the following statement:

If $0 = 1$, then $1 = 2$.

Since the hypothesis of this statement is false, the entire statement – as a whole – is true.

Such statements are called vacuously true (or true by default).

Example 1

Use truth tables to show that

$p \to q \equiv \sim p \vee q$

p	q	p → q	~p ∨ q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 2

Use De Morgan's Laws and the result of Example 1 to show that

$\sim(p \to q) \equiv p \wedge \sim q$

$\sim(\sim p \vee q)$

$p \wedge \sim q$

The Negation of Conditional Statements

The negation of "if p then q" is logically equivalent to "p and not q."

Warning!

The negation of an if-then statement does not start with the word if!

Contrapositive vs. Converse

The contrapositive of $p \to q$ is $\sim q \to \sim p$.

A conditional statement and its contrapositive are logically equivalent.

The converse of $p \to q$ is $q \to p$.

A conditional statement and its converse are not logically equivalent (although, in some cases, the converse may be true when the conditional is true).

Inverse

The inverse of $p \to q$ is $\sim p \to \sim q$.

A conditional statement and its inverse are not logically equivalent.

The converse and inverse of a conditional statement are logically equivalent to each other.

3

Example 3

Write the contrapositive, converse, and inverse of the following statement:

If today is Labor Day, then yesterday was Sunday.

l - today is labor day

s - yesterday was sunday

$l \to s$

$\sim s \to \sim l$

$s \to l$

$\sim l \to \sim s$

Example 4

If possible, give an example of a true conditional sentence for which

a) The converse is true.

b) The converse is false.

c) The contrapositive is true.

d) The contrapositive is false.

if A then A

if squares then rectangle

Equivalent

None

Example 5

Show that a conditional statement is not logically equivalent to its converse.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

4

Only If

If p and q are statement variables, p only if q means "if not q then not p" or, equivalently (by contraposition) "if p then q."

Biconditional Statements

If p and q are statement variables, the biconditional of p and q is $p \leftrightarrow q$, read "p if and only if q" and sometimes abbreviated as "p iff q."

$p \leftrightarrow q$ is true if p and q have the same truth value and false if p and q have opposite truth values.

Biconditional Statements

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

5

Example 5

Use truth tables to show that
$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

p	q	$p \leftrightarrow q$	$\sim p \vee q$	$\sim q \vee p$	$(\sim p \vee q) \wedge (\sim q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Order of Operations for Logical Operators

1. Evaluate negations first.
2. Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

Example 6

Determine whether each of the following is a tautology, a contradiction, or neither.

a) $((p \rightarrow q) \rightarrow p) \rightarrow p$
b) $p \wedge (p \leftrightarrow q) \wedge \sim q$

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow p$	$((p \rightarrow q) \rightarrow p) \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

p	q	$p \leftrightarrow q$	
T	T	T	F
T	F	F	F
F	T	F	F
F	F	T	F

6

Necessary and Sufficient Conditions

Let r and s be statements.

r is a sufficient condition for s means $r \rightarrow s$.

r is a necessary condition for s means $\sim r \rightarrow \sim s$.

r is a necessary and sufficient condition for s means $r \leftrightarrow s$.

not error ness. correct

error → not c

c → not err

Clarifying Remarks

In logic, a hypothesis and conclusion are not required to have related subject matters.

Simple example:
If Albert Pujols is a baseball player, then Brett Favre is a football player.

Simple example:

If Albert Pujols is a baseball player, then Brett Favre is a football player.

Clarifying Remarks

In informal language, simple conditional statements are often interpreted as biconditionals.

Simple example:

If you eat your dinner, you will get dessert.

7

Valid and Invalid Arguments

Wednesday, August 31, 2022 2:00 PM

CS1200+Lecture+5+H...

Section 2.3

Valid and Invalid Arguments

Arguments

An argument is a sequence of statements, and an argument form is a sequence of statement forms.

All statements in an argument or argument form, except for the final one, are called premises (or assumptions or hypotheses). The final statement is called the conclusion (typically following "therefore").

Arguments

An argument form is said to be valid if there is no possible way for the conclusion to be false when all premises of the argument form are true.

An argument is valid whenever its form is valid.

1

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all premises.
3. Identify all rows of the truth table in which all premises are true. These rows are called <u>critical rows</u>.
4. Determine the truth value of the conclusion for all critical rows.

Testing an Argument Form for Validity

5. If there is a critical row in which the conclusion is false, the argument form is invalid. If the conclusion in every critical row is true, then the argument form is valid.

Example 1

Use a truth table to test the argument for validity.

p
$p \to q$
$\sim q \vee r$
$\therefore r$

p
p → q
~q ∨ r
r

p	q	r	~q	p → q	~q ∨ r	r
T	T	T	F	T	T	T
T	T	F	F	T	F	
T	F	T	T	F	T	
T	F	F	T	F	T	
F	T	T	F	T	T	
F	T	F	F	T	F	
F	F	T	T	T	T	
F	F	F	T	T	T	

← not false, so valid

2

Example 2

Use a truth table to test the argument for validity.

$p \vee q$
$p \to \sim q$
$p \to r$
$\therefore r$

p	q	r	$p \vee q$	$p \to \sim q$	$p \to r$	r
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Invalid

Syllogisms

A syllogism is an argument form consisting of two premises and a conclusion.

The first premise is called the major premise.
The second premise is called the minor premise.

Famous Syllogisms

Modus Ponens
If p then q.
p
$\therefore q$

Modus Tollens
If p then q.
$\neg q$
$\therefore \neg p$

Example 3

Recall that an argument is valid whenever its form is valid. Is the following argument valid?

If smoking is healthy, then my physician will tell me to smoke.

Smoking is healthy.

Therefore, my physician will tell me to smoke.

Example 4

Use modus ponens or modus tollens to fill in the blank in the following argument so that it becomes a valid inference.

If logic is easy, then you would not need this class.

You need this class.

Therefore, ___________.

Rules of Inference

A rule of inference is a form of argument that is valid.

Examples:

- Modus Ponens
- Modus Tollens

Generalization

The following argument forms are valid.

p
$\therefore p \vee q$

q
$\therefore p \vee q$

Specialization

The following argument forms are valid.

$p \wedge q$
$\therefore p$

$p \wedge q$
$\therefore q$

Elimination

The following argument forms are valid.

$p \vee q$
$\sim q$
$\therefore p$

$p \vee q$
$\sim p$
$\therefore q$

5

Transitivity

The following argument form is valid.

$p \to q$
$q \to r$
$\therefore p \to r$

Proof by Division into Cases

The following argument form is valid.

$p \vee q$
$p \to r$
$q \to r$
$\therefore r$

Fallacies

A fallacy is an error in reasoning that results in an invalid argument.

Common fallacies:

- Using ambiguous premises
- Circular reasoning (assuming what you're trying to prove)
- Jumping to a conclusion
- Converse Error
- Inverse Error

Converse Error

The following argument form is not valid.

$p \rightarrow q$
q
$\therefore p$

Inverse Error

The following argument form is not valid.

$p \rightarrow q$
$\sim p$
$\therefore \sim q$

Example 5

Determine whether the argument is valid, exhibits the converse error, or exhibits the inverse error.

If Jules solved this problem correctly, then Jules obtained the answer 2.
Jules obtained the answer 2.
∴ Jules solved this problem correctly.

Example 6

Determine whether the argument is valid, exhibits the converse error, or exhibits the inverse error.

If I play too many games, I won't finish my homework.

If I don't finish my homework, I won't do well on the exam tomorrow.

∴ If I play too many games, I won't do well on the exam tomorrow.

Example 7

Determine whether the argument is valid, exhibits the converse error, or exhibits the inverse error.

If this number is larger than 2, then its square is larger than 4.

This number is not larger than 2.

∴ The square of this number is not larger than 4.

Sound Arguments

An argument is called sound iff it is valid and all its premises are true. An argument that is not sound is called unsound.

Remember that validity is a property of an argument form.

We can only be sure that the conclusion of an argument is true when we know the argument is sound.

8

Example 3 (revisited)

The following argument is valid. Is it sound?

If smoking is healthy, then my physician will tell me to smoke.

Smoking is healthy.

Therefore, my physician will tell me to smoke.

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Example 8

Show that the contradiction rule is valid by showing the following argument form is valid.

$\sim p \to c$ where c is a contradiction.

$\therefore p$

p	c	$\sim p \to c$	p
T	F	T	T
F	F	F	F

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Predicates and Quantified Statements I

Friday, September 2, 2022 1:57 PM

CS1200+Lecture+6+H...

New Room: Bertelsmeyer B-10

Section 3.1

Predicates and
Quantified Statements I

Predicates

A predicate is a sentence which contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The domain of a predicate variable is the set of all values that may be substituted in place of the variable.

Predicates

If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x.

The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}$$

1

Example 1

Let $B(x)$ be the predicate "$-10 \leq x < 10$."
Find the truth set of $B(x)$ for each of the following domains.

a) $\mathbb{Z}$
b) $\mathbb{Z}^+$
c) The set of all even integers.

Quantifiers

Quantifiers are words that refer to quantities such as "some" or "all" and indicate for how many elements a given predicate is true.

Quantifiers provide one way to obtain statements from predicates.

The Universal Quantifier

$\forall$ is the universal quantifier
$\forall$ is typically read as "for all."

Other typical readings:
"for every," "for each," "for any," "given any"

Universal Statements

Let $Q(x)$ be a predicate and D the domain of x. A universal statement is a statement of the form

$$\forall x \in D, Q(x)$$

It is defined to be true if and only if $Q(x)$ is true for each individual $x \in D$.

A value x for which $Q(x)$ is false is called a counterexample to the universal statement.

Example 2

Which of the following are equivalent ways of expressing the statement

$\forall n \in \mathbb{Z}$, if n^2 is even then n is even.

a) All integers have even squares and are even. ✗

b) Given any integer whose square is even, that integer is itself even. ✓

c) For all integers, there are some whose square is even. ✗

Example 2

Which of the following are equivalent ways of expressing the statement

$\forall n \in \mathbb{Z}$, if n^2 is even then n is even.

d) Any integer with an even square is even. ✓

e) If the square of an integer is even, then that integer is even. ✓

f) All even integers have even squares. ✗

3

Example 3

Find a counterexample to show that the statement

$$\forall x, y \in \mathbb{R}, \sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

is false.

The Existential Quantifier

∃ is the existential quantifier

∃ is typically read as "there exists."

Other typical readings:

"there is a," "we can find a," "for some," "there is at least one," "for at least one"

Existential Statements

Let $Q(x)$ be a predicate and D the domain of x. An existential statement is a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

It is defined to be true if and only if $Q(x)$ is true for at least one $x \in D$.

Example 4

Which of the following are equivalent ways of expressing the statement

$$\exists x \in \mathbb{R} \text{ such that } x^2 = 2$$

a) The square of each real number is 2. ✗

b) Some real numbers have square 2. ✓

c) The number x has square 2, for some real number x. ✓

Example 4

Which of the following are equivalent ways of expressing the statement

$$\exists x \in \mathbb{R} \text{ such that } x^2 = 2$$

d) If x is a real number, then $x^2 = 2$. ✗

e) Some real number has square 2. ✓

f) There is at least one real number whose square is 2. ✓

Formal vs. Informal Language

There is often more than one way we can informally state a formal statement.

Formally, we want universal and existential quantifiers at the beginning of a sentence. Informally, we often place them at the end.

Example 5

Rewrite each statement so that the quantifier trails the rest of the sentence.

a) For any isosceles triangle T, two angles of T are equal.

b) There exists a continuous function f such that f is not differentiable.

Universal Conditional Statements

One particularly important statement form is the universal conditional statement

$\exists x$, if $P(x)$ then $Q(x)$

Example 6

Rewrite the statement

Some questions are easy

in the following two forms:

1) $\exists$ questions x such that x is easy
2) $\exists x$ such that x is a question and x is easy

6

Example 7

Rewrite the statement

Every computer science student needs to take data structures

in the following two forms:

1) $\forall x$, if x CS then x take data structures
2) $\forall$ CS x, x take data structures (without an if-then)

Implicit Quantification

Often, the universality or existentiality of a statement is implied, not explicitly written.

Example 8

Rewrite the statement as either an explicitly existential or explicitly universal statement.

a) The sum of the angles of a triangle is 180°.
b) The number 12 is divisible by at least two primes.

For all triangles T, the sum of the angles of T is 180°.

$\Rightarrow$ and $\Leftrightarrow$

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$ or, equivalently, $\forall x, P(x) \rightarrow Q(x)$.

The notation $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

8

Predicates and Quantified Statements II

Wednesday, September 7, 2022 2:00 PM

CS1200+Lecture+7+H...

Section 3.2

Predicates and Quantified Statements II

Negation of a Universal Statement

The negation of a statement of the form

$\forall x \in D, Q(x)$

is logically equivalent to a statement of the form

$\exists x \in D$ such that $\sim Q(x)$.

Negation of an Existential Statement

The negation of a statement of the form

$\exists x \in D$ such that $Q(x)$

is logically equivalent to a statement of the form

$\forall x \in D, \sim Q(x)$.

1

Example 1

Write formal and informal negations of the following statements.

a) $\exists$ a movie m such that m is over 6 hours long.

b) All real numbers are positive, negative, or zero.

Negation of a Universal Conditional

The negation of a statement of the form

$\forall x$, if $P(x)$ then $Q(x)$

is logically equivalent to a statement of the form

$\exists x$ such that $P(x)$ and $\sim Q(x)$.

Example 2

Write a formal and an informal negation for the following statement.

$\forall n \in \mathbb{Z}$, if n is prime then n is odd or $n = 2$.

2

Example 3

Determine whether the proposed negation is correct. If it is not, write a correct negation.

Statement: For every integer n, if n^2 is even then n is even.

Proposed negation: For every integer n, if n^2 is even then n is not even.

Vacuous Truth of Universal Statements

In general, a statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

is called vacuously true (or true by default) iff $P(x)$ is false for every x in D.

"In General"

In ordinary language, the words "in general" mean that something is usually, but not always, the case.

In mathematics, the words "in general" mean that something is always true.

3

Section 3.3

Statements with Multiple Quantifiers

Statements with Multiple Quantifiers

When a statement contains more than one kind of quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.

Statements with Multiple Quantifiers

For a statement of the form

$\forall x \in D, \exists y \in E$ such that x and y satisfy $P(x, y)$

to be true, you must be able to meet this challenge:

1. Imagine that someone is allowed to choose any element from D and give it to you. Call it x.
2. Now that you have x, your challenge is to find an element $y \in E$ so that the chosen x and found y together satisfy the property $P(x, y)$.

Note that you do not have to select y until after x is chosen, allowing you to pick a different y for each x.

Statements with Multiple Quantifiers

For a statement of the form

$\exists x \in D$ such that $\forall y \in E$, x and y satisfy $P(x, y)$

to be true, you must be able to must find one single element (call it x) in D which meets the following challenge:

1. After you have selected x, someone is allowed to choose any element whatsoever from E and give it to you. Call it y.
2. Without changing your x, you must show that the chosen y and pre-selected x together satisfy the property $P(x, y)$.

Negation with Multiple Quantifiers

The negation of

$\forall x \in D, \exists y \in E$ such that x and y satisfy $P(x, y)$

is

$\exists x \in D$ such that $\forall y \in E$, x and y satisfy $\sim P(x, y)$

The negation of

$\exists x \in D$ such that $\forall y \in E$, x and y satisfy $P(x, y)$

is

$\forall x \in D, \exists y \in E$ such that x and y satisfy $\sim P(x, y)$

Example 4

Rewrite the statement in English as simply as possible, and write a negation for the statement.

$\exists u \in \mathbb{R}$ such that $\forall v \in \mathbb{R}$, $uv = v$

Example 5

Rewrite the statement formally using quantifiers and variables. Then, write a negation for the statement.

There is a program that gives the correct answer to every question which is posed to it.

Changing the Order of Quantifiers

In a statement containing both ∀ and ∃, changing the order of the quantifiers can significantly change the meaning of the statement.

If one quantifier immediately follows another of the same type (i.e. both are ∀ or both are ∃), then the order does not affect the meaning.

Example 6

Write a new statement by changing the order of the quantifiers. Identify which statement is true: the original statement, the version with interchanged quantifiers, neither, or both.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x < y.$$

Then, rewrite the statement in English.

Arguments with Quantified Statements

Friday, September 9, 2022 1:58 PM

CS1200+Lecture+8+H...

Section 3.4

Arguments with Quantified Statements

Universal Instantiation

If a property is true of everything in a set, then it is true of any particular thing in the set.

Recall: Famous Arguments

Modus Ponens

If p then q.
p
$\therefore q$

Modus Tollens

If p then q.
$\neg q$
$\therefore \neg p$

1

Universal Modus Ponens

Formal version:

$\forall x$ if $P(x)$ then $Q(x)$.

$P(a)$ for a particular a

$\therefore Q(a)$

Informal version:

If x makes $P(x)$ true, then x makes $Q(x)$ true.

a makes $P(x)$ true.

$\therefore$ a makes $Q(x)$ true.

Example 1

Use universal modus ponens to fill in a valid conclusion for the argument.

If an integer n equals $2k$ and k is an integer, then n is even.

0 equals $2 \cdot 0$ and 0 is an integer.

$\therefore$ ________________.

Universal Modus Ponens in a Proof

Goal: Prove that the sum of any two even integers is even.

Background definition:

An integer is even iff it equals twice some integer.

2

Universal Modus Ponens in a Proof

1. Suppose m and n are particular but arbitrarily chosen even integers.
2. Then, $m = 2r$ for some integer r and $n = 2s$ for some integer s.
3. Hence,
$$m + n = 2r + 2s$$
$$= 2(r + s)$$
4. Since $(r + s)$ is an integer, $2(r + s)$ is even.
5. Thus, $m + n$ is even.

Universal Modus Ponens in a Proof

Where did we use Universal Modus Ponens?

Step 2:

If an integer is even, then it equals twice some integer.

m is a particular even integer.

$\therefore$ m equals twice some integer, say r.

(Similar argument with n and s.)

Universal Modus Ponens in a Proof

Where did we use Universal Modus Ponens?

Step 4:

For all u and v, if u and v are integers, then $u + v$ is an integer.

r and s are two particular integers.

$\therefore$ $r + s$ is an integer

Universal Modus Ponens in a Proof

Where did we use Universal Modus Ponens?

We also used it on step 3, and we used it a second time on step 4.

Universal Modus Tollens

Formal version:

$\forall x$ if $P(x)$ then $Q(x)$.

$-Q(a)$ for a particular a

$\therefore -P(a)$

Informal version:

If x makes $P(x)$ true, then x makes $Q(x)$ true.

a does not make $Q(x)$ true.

$\therefore$ a does not make $P(x)$ true.

Example 2

Use universal modus tollens to fill in a valid conclusion for the argument.

All irrational numbers are real numbers.

$\frac{1}{0}$ is not a real number.

$\therefore$ ________________.

Recall: Valid and Sound Arguments

An argument form is said to be valid if there is no possible way for the conclusion to be false when all premises of the argument form are true.

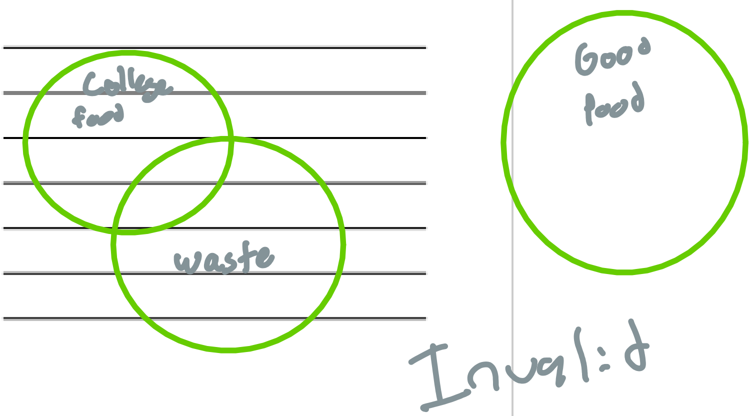
An argument is called valid iff its form is valid.

An argument is called sound iff its form is valid and its premises are true.

Example 3

Indicate whether the argument is valid or invalid. Support your answer by drawing a diagram.

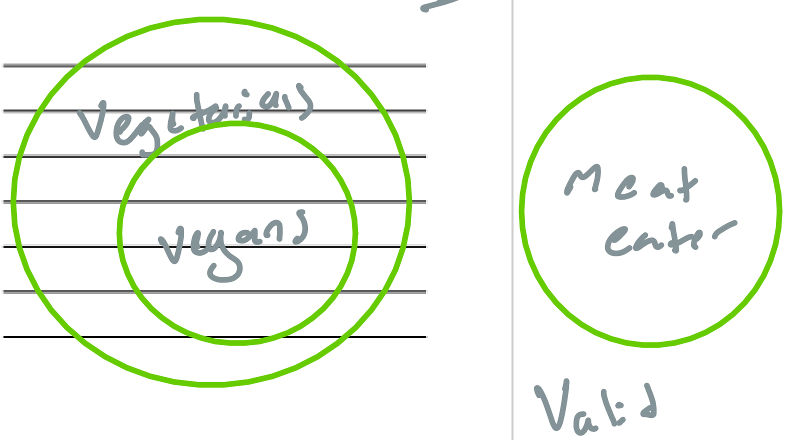
No college cafeteria food is good.
No good food is wasted.
∴ No college cafeteria food is wasted.



Example 4

Indicate whether the argument is valid or invalid. Support your answer by drawing a diagram.

No vegetarians eat meat.
All vegans are vegetarian.
∴ No vegans eat meat.



5

Converse Error (quantified form)

Formal version:

$\forall x$ if $P(x)$ then $Q(x)$.

$Q(a)$ for a particular a

$\therefore P(a)$

Informal version:

If x makes $P(x)$ true, then x makes $Q(x)$ true.

a makes $Q(x)$ true.

$\therefore$ a makes $P(x)$ true.

Note that the converse error has an invalid conclusion.

Inverse Error (quantified form)

Formal version:

$\forall x$ if $P(x)$ then $Q(x)$.

$\sim P(a)$ for a particular a

$\therefore \sim Q(a)$

Informal version:

If x makes $P(x)$ true, then x makes $Q(x)$ true.

a does not make $P(x)$ true.

$\therefore$ a does not make $Q(x)$ true

Note that the inverse error has an invalid conclusion.

Example 5

Determine whether the argument is valid or invalid.

If a graph has no edges, then it has a vertex of degree zero.

This graph has at least one edge.

$\therefore$ This graph does not have a vertex of degree zero.

Invalid

Inverse

6

Example 6

Determine whether the argument is valid or invalid.

For every student x, if x studies discrete math, then x is good at logic.

Susan studies discrete mathematics.

∴ Susan is good at logic.

Valid

Example 7

Determine whether the argument is valid or invalid.

All cheaters sit in the back row.

Monty sits in the back row.

∴ Monty is a cheater.

Invalid

Converse

Example 8

Determine whether the argument is valid or invalid.

All students who failed Prof. Simpson's class are in a fraternity.

Phillip is in a fraternity.

∴ Phillip failed Prof. Simpson's class.

Invalid

Converse

Converse Errors in Real Life

Suppose Abigail believes the following:

For every x, if x has Covid-19, x has a fever, cough, and shortness of breath.

Then, Abigail sits next to Steve in class, and Steve is coughing and breathing heavily.

Abigail decides Steve must have Covid-19.

Converse Errors in Real Life

Suppose Dr. Smith knows the following:

For every x, if x has Covid-19, x has a fever, cough, and shortness of breath.

Then, a patient comes to Dr. Smith's office with a fever, cough, and is breathing heavily.

Dr. Smith suspects the patient has Covid-19 and then performs appropriate testing.

This form of reasoning is sometimes called abduction.

Direct Proof And Counterexample I

Monday, September 12, 2022 2:01 PM

CS1200+Lecture+9+H...

Section 4.1

Direct Proof and Counterexample I: Introduction

Why Is Proof Important?

Writing a proof forces us to become aware of weaknesses in our arguments and in the unconscious assumptions we have made.

In a proof, we must say exactly what we mean and mean exactly what we say!

Are We Doing Insanely Hard Proofs?

No!

Most proofs we do will assume you are familiar with the basic laws of algebra.

1

Even and Odd Integers

An integer n is even iff $n = 2k$ for some integer k.

An integer n is odd iff $n = 2k + 1$ for some integer k.

Example 1

Prove that -403 is odd.

Prime and Composite Integers

An integer $n > 1$ is prime iff for all positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

An integer $n > 1$ is composite iff there exist positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

Proving Existential Statements

A statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is true iff $Q(x)$ is true for at least one x in D.

Constructive proof method #1:
Find an x in D that makes $Q(x)$ true.

Perfect Squares

An integer n is called a perfect square iff $n = k^2$ for some integer k.

Example 2

Prove that there is a perfect square that can be written as the sum of two other perfect squares.

3

Proving Existential Statements

A statement of the form

$\exists x \in D$ such that $Q(x)$

is true iff $Q(x)$ is true for at least one x in D.

Constructive proof method #2:

Give a set of directions for how to find an x in D that makes $Q(x)$ true.

Example 3

Prove that there are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer

Proving Existential Statements

A statement of the form

$\exists x \in D$ such that $Q(x)$

is true iff $Q(x)$ is true for at least one x in D.

Nonconstructive proof method #1:

Show that the existence of a value x that makes $Q(x)$ true is guaranteed by an axiom or previously proven theorem.

4

Proving Existential Statements

A statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is true iff $Q(x)$ is true for at least one x in D.

Nonconstructive proof method #2:
Show that the assumption that no such value of x exists leads logically to a contradiction.

Disproving Universal Statements

To disprove a statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

find a value of $x \in D$ for which $P(x)$ is true and $Q(x)$ is false. Such an x is called a counterexample.

Example 4

Disprove the statement by finding a counterexample.

For all integers m and n, if $2m + n$ is odd then m and n are both odd.

Proving Universal Statements

To prove a statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

it may be possible to check the statement for every possible element x. This will only work when either D is a finite (and relatively small) set or when only a finite (and relatively small) number of elements satisfy $P(x)$.

Example 5

Prove the statement.

For each integer n with $1 \le n \le 10$, $n^2 - n + 11$ is a prime number.

Generalizing from the Generic Particular

To show that every element of a set satisfies a certain property, suppose x is a particular but arbitrarily chosen element of the set and show that x satisfies the property.

This is the basis for the method of direct proof.

The Method of Direct Proof

1. Express the statement to be proved in the form
   $\forall x \in D$, if $P(x)$ then $Q(x)$
2. Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true.
3. Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules of logical inference.

Example 6

Prove the following theorem.

The sum of any two odd integers is even.

Existential Instantiation

If the existence of a certain kind of object is assumed or has been deduced, then it can be given a name provided that name is not currently being used to refer to something else in the same discussion.

7

Example 7

Prove the following theorem.

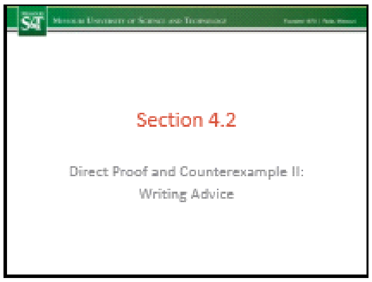
Whenever n is an odd integer, $5n^2 + 7$ is even.

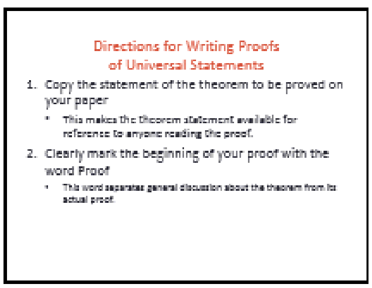
8

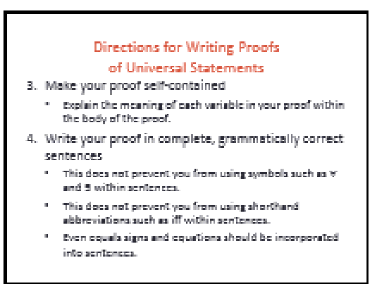
Direct Proof and Counterexample II

Wednesday, September 14, 2022 1:59 PM

CS1200+Lecture+10+...







1

Directions for Writing Proofs of Universal Statements

5. Keep your reader informed about the status of each statement in your proof
   - Make sure it is clear to the reader whether something in the proof has already been established, is assumed, or is still to be deduced.
6. Give a reason for each assertion in your proof
   - Every assertion should come directly from a hypothesis of the theorem being proved, follow from the definition of a term, be a result obtained earlier in the proof, or be a mathematical result that has previously been established.
   - The specific reason should be clearly stated.

Directions for Writing Proofs of Universal Statements

7. Include "little words and phrases" that make the logic of your arguments clear
   - Starting a sentence with *because* or *since* and giving the reason immediately is sometimes preferable to starting a sentence with *so*, *Then*, *Thus*, *hence*, or *therefore* and stating the reason at the end of the sentence.
   - If a sentence expresses a new thought or fact that does not follow as an immediate consequence of a preceding statement, using *observe that*, *recall that*, or *note that* might be useful.

Directions for Writing Proofs of Universal Statements

8. Display equations and inequalities
   - We typically prefer placing equations and inequalities on separate lines to increase readability.
9. Make the end of the proof clear to the reader
   - Traditional: Q.E.D.
     Latin abbreviation for *quod erat demonstratum*, meaning *which was to be demonstrated*.
   - In print: ■

Variations Among Proofs

It is exceedingly rare for two proofs of an identical statement to be identical when written by two different people.

This does not mean one of them is wrong.

This also does not mean both of them are right.

Example 1

Prove the statement. Use only the definitions of the terms. Do not use any previously established properties of even/odd integers in your proof.

The difference of any even integer minus any odd integer is odd.

Common Mistakes

1. Arguing from examples
   - Just because something is true in one case or a handful of cases does not mean it is true in general.
2. Using the same letter to mean two different things
   - For example, if you want to work with two distinct odd integers m and n, they shouldn't both equal $2k + 1$. If they do, they're not distinct.

3

Common Mistakes

3. Jumping to a conclusion
   - Do not allege that something is true without giving an adequate reason.
4. Assuming what is to be proved
   - This is sometimes subtle, but it's very bad.
5. Confusion between what is known and what is still to be shown
   - If you want to state – in advance – what your proof aims to show, write "We aim to show…" or something like that. Then, actually show it!

Common Mistakes

6. Use of *any* when the correct word is *some*
   - These words are occasionally – but not always – interchangeable.
7. Misuse of the word *if*
   - We often use *if* instead of *because*. This is bad.

Example 2

Find the mistake(s) in the "proof" of the statement.

Statement: The sum of any two even integers equals $4k$ for some integer k.

"Proof": Suppose m and n are any two even integers. By the definition of even, $m = 2k$ for some integer k and $n = 2k$ for some integer k. By substitution, $m + n = 2k + 2k = 4k$. This is what was to be shown.

4

Example 2 (continued)

Is the statement true or false?

Statement: The sum of any two even integers equals $4k$ for some integer k.

Proving an Existential Statement is False

To prove an existential statement is false, you must prove its negation (which is a universal statement) is true.

Example 3

Prove that the statement is false.

Statement: There exists an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

Let k be an integer where $k \geq 4$.

Then, $2k^2 - 5k + 2 = (2k-1)(k-2)$ by factoring

$m = 2k-1$ and $n = k-2$ are both integers and since $k \geq 4$, neither m or $n = 1$

Therefore, $2k^2 - 5k + 2 = mn$, meaning there are two factors other than one.

By definition, $2k^2 - 5k + 2$ is composite. Done

Example 4

Determine whether the statement is true or false. Justify your answer with a proof or counterexample.

The product of any even integer and any integer is even.

m is any even integer, n is any integer.

m = 2k for some integer k

m • n = (2k)n by substitution

= 2(kn) by associativity of multiplication

Since the product of two integers is an integer, q = kn = integer

therefore mn = 2q and m is even.

Done

Example 5

Determine whether the statement is true or false. Justify your answer with a proof or counterexample.

For all integers a, b, and c, if a, b, and c are consecutive, then $a + b + c$ is even

$2 + 3 + 4 = 9$

9 is odd

Example 6

Determine whether the statement is true or false. Justify your answer with a proof or counterexample.

Any product of four consecutive integers is one less than a perfect square.

$n(n+1)(n+2)(n+3)$

$= (n^2+n)(n^2+5n+6)$

$= n^4+5n^3+6n^2+n^3+5n^2+6n$

$= n^4+6n^3+11n^2+6n$

6

Direct Proof and Counterexample III

Monday, September 19, 2022 2:04 PM

CS1200+Lecture+12+...

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Section 4.3

Direct Proof and Counterexample III: Rational Numbers

Rational Numbers

A real number r is rational iff it can be expressed as the quotient of two integers with a nonzero denominator.

A real number that is not rational is called irrational.

Example 1

Determine whether each number is rational. If so, write it as the ratio of two integers.

a) $-\frac{22}{7}$

b) 1.2345

c) 0.454545 ...

1

$.73\overline{73} \cdot 100 = 73.\overline{73} - \text{orig} = 73$

$100x - x = 99x$

$99x = 73$

$x = \frac{73}{99}$

52474.17

$- \; 52.474\overline{417}$

$10000x - x = 52469 1.7$

$9999x = 524691.7$

The Zero Product Property

The product of two real numbers is zero iff at least one of the real numbers is zero.

Example 2

If m and n are integers and neither m nor n is zero, is

$$\frac{m+n}{mn}$$

a rational number?

Example 3

Prove that every integer is a rational number.

2

Theorem

Every integer is a rational number.

Example 4

Prove that the sum of two rational numbers is rational.

Theorem

The sum of any two rational numbers is rational.

3

Example 5

When expressions of the form $(x-r)(x-s)$ are multiplied out, a quadratic polynomial is obtained. What can be said about the coefficients of the polynomial obtained in each of these scenarios:

i. Both r and s are odd
ii. Both r and s are even
iii. One is even and the other is odd

Example 5 (continued)

Use the results of this example to explain why

$$x^2 - 1253x + 255$$

cannot be written as a product of the form

$$(x-r)(x-s)$$

IV Divisibility

Monday, September 26, 2022 1:59 PM

CS1200+Lecture+13+...

Section 4.4

Direct Proof and Counterexample IV: Divisibility

Divisibility

If n and d are integers, then n is divisible by d iff n equals d times some integer and $d \neq 0$.

Instead of saying n is divisible by d, we can say

- n is a multiple of d
- d is a factor of n
- d is a divisor of n
- d divides n

Divisibility

The notation $d|n$ is read "d divides n."

Symbolically,

$d|n$ iff $\exists k \in \mathbb{Z}$ such that $n = dk$ and $d \neq 0$.

The notation $d \nmid n$ is read "d does not divide n."

1

Example 1

Does 12 divide 60? Justify your answer.

Theorem

For all integers a and b, if a and b are positive and a divides b, then $a \leq b$.

Corollary

For all integers a and positive integers b, if a divides b, then $|a| \leq b$.

Example 2

Prove that the only divisors of 1 are 1 and -1.

Example 3

Prove that divisibility is transitive.

That is, prove that if $a|b$ and $b|c$ then $a|c$.

Example 4

Disprove the following statement:

For all integers a and b, if $a|b$ and $b|a$ then $a = b$.

The Unique Factorization of Integers

Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers $p_1, \dots, p_k$, and positive integers $e_1, \dots, e_k$ such that

$$n = p_1^{e_1} \cdots p_k^{e_k}$$

and any other expression for n as a product of primes is identical to this except possibly for the order in which the factors are written.

In standard factored form, the primes are written in ascending order.

Example 5

Write the integer 24948 in standard factored form.

Example 6

Determine whether the statement is true or false. Prove (if true) or give a counterexample (if false).

The sum of any three consecutive integers is divisible by 3.

Section 4.5

Direct Proof and Counterexample V:
The Quotient-Remainder Theorem

24948 = 2 · 12474
12474 = 2 · 6237
6237 = 3 · 2079
2079 = 3 · 693
693 = 3 · 231
231 = 3 · 77
77 = 7 · 11

$2^2\, 3^4\, (7)(11)$

4116 = 2 · 2058
2058 = 2 · 1029
1029 = 3 · 343
343 = 7 · 49
49 = 7 · 7

1617 = 3 · 539
539 = 7 · 77
77 = 7 · 11

2205 = 5 · 441
441 = 3 · 147
147 = 3 · 49
49 = 7 · 7

The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.

q is sometimes called $n\ div\ d$.

r is sometimes called $n\ mod\ d$.

$b = 12m + 5$

$76 = 12q + r$

$r = 11$

$17(7) = 12q + 11$

$17 \quad 1$

Example 7

For each of the following values of n and d, find integers q and r such that $n = dq + r$ and $0 \leq r < d$.

a) $n = 67, d = 4$
b) $n = -67, d = 4$

Example 8

Evaluate the expression.

$50\ mod\ 7$

Example 9

Suppose a is any integer. If $a \ mod \ 7 = 4$, what is $5a \ mod \ 7$?

6

Division into Cases

Wednesday, September 28, 2022 2:02 PM

CS1200+Lecture+14+...

Section 4.5

Direct Proof and Counterexample V: Division Into Cases

Absolute Value

For any real number x, the absolute value of x, denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Lemma 1

For every real number r, $-|r| \leq r \leq |r|$.

1

Lemma 2

For every real number r, $|-r| = |r|$.

The Triangle Inequality

For all real numbers x and y, $|x + y| \leq |x| + |y|$.

Example 1

Prove the Triangle Inequality.

Example 2

Prove the following statement.

The fourth power of any integer has the form $8m$ or $8m + 1$ for some integer m.

2

Section 4.7

Indirect Argument:
Contradiction and Contraposition

Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Example 3

Prove the statement.

For any integer n, $n^2 - 2$ is not divisible by 4.

Case 1: n is odd

$n = 2k+1$ for some integer k

assume n^2-2 is divisible by 4

$n^2-2 = 4r$ for some integer r

$n^2-2 = (2k+1)^2 - 2 = 4k^2 + 4k + 1 - 2$

$= 4k^2 + 4k - 1$

$\neq 4r$

Contradiction, thus n^2-2 is not divisible by 4

Case 2: n is even

$n = 2k$ for some integer k

Assume $n^2-2 = 4r$ some integer r

$n^2-2 = (2k)^2 - 2 = 4k^2 - 2 \neq 4r$

by contradiction, n^2-2 is not divisible by 4

by division by cases, ∎

3

Proof by Contraposition

1. Express the statement to be proved in the form
   $\forall x$ in D, if $P(x)$ then $Q(x)$.
2. Rewrite the statement in the contrapositive form
   $\forall x$ in D, if $\sim Q(x)$ then $\sim P(x)$.
3. Prove the contrapositive by a direct proof.

Example 4

Prove the statement.

For every integer n, if n^2 is odd then n is odd.

The Relationship between Contradiction and Contraposition

To prove

$\forall x$ in D, if $P(x)$ then $Q(x)$

by contradiction instead of contraposition, you can suppose

$\exists x$ in in D such that $P(x)$ and $\sim Q(x)$

and arrive at the contradiction that both $P(x)$ and $\sim P(x)$.

4

Indirect Argument

Monday, October 3, 2022 1:59 PM

CS1200+Lecture+15+...

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Section 4.8

Indirect Argument:
Two Famous Theorems

Two Famous Theorems

This section looks at two famous theorems.

We will state both theorems, plus a proposition necessary to prove the second.

We will work through the proof of one of the two theorems in detail.

First Famous Theorem

$\sqrt{2}$ is irrational.

1

Proposition

For any integer a and prime number p, if $p|a$ then $p \nmid (a+1)$.

Second Famous Theorem

The set of prime numbers is infinite.

Example 1

Prove the First Famous Theorem or, alternately, prove the proposition and the Second Famous Theorem.

Section 4.10

Application: Algorithms

Algorithms

An algorithm is a step-by-step method for performing some action.

Variables

When working with high-level programming languages, a variable can be used to refer to a location in a computer's memory and/or the contents of that location.

The data type of a variable indicates the set from which the variable takes its values.

Variables

An assignment statement gives a value to a variable. It has the form

$$x := e$$

where x is a variable and e is an expression.

Such a statement is read "x is assigned the value e" or "x is defined to equal e."

Iterative Processes and Trace Tables

The values of the variable(s) involved in an iterative process (loop) within an algorithm should be tracked carefully.

This can be done using a trace table.

Example 2

Find the values of e and f after execution of the following loop by first making a trace table.

$e := 2, f := 0$
for $k = 1$ to 3
  $e := e \cdot k$
  $f := e + f$
next k

4

Formal Descriptions of Algorithms

When formally describing an algorithm, we generally include the following information:

1. The name of the algorithm.
2. A brief description of how the algorithm works.
3. The input variable names, labeled by data type.
4. The statements that make up the body of the algorithm, possibly with explanatory comments.
5. The output variable names, labeled by data type.

The Division Algorithm

Input: a (a nonnegative integer) and d (a positive integer)

Algorithm body:

$r := a, q := 0$

while $r \geq d$

$r := r - d$

$q := q + 1$

end while

(After the loop, $a = dq + r$.)

Output: q, r (nonnegative integers)

Example 3

Make a trace table to trace the action of the division algorithm for the input variables $a = 26$ and $d = 7$.

Algorithms and Sequences

Monday, October 3, 2022 1:59 PM

CS1200+Lecture+16+...

Section 4.10

Application: Algorithms

The Greatest Common Divisor

Let a and b be integers that are not both zero. The greatest common divisor of a and b, denoted $gcd(a, b)$, is the integer d with the following properties:

1. d divides both a and b.
2. For every integer c, if $c|a$ and $c|b$, then $c \leq d$.

Lemma

If a and b are integers that are not both zero and if q and r are any integers such that

$$a = bq + r$$

then

$$gcd(a, b) = gcd(b, r)$$

1

The Euclidean Algorithm

Goal: Find $gcd(A, B)$ for integers $A > B \geq 0$.

Procedure:

If $B = 0$, then $gcd(A, B) = A$.

If $B \neq 0$, then divide A by B using the division algorithm, obtaining a quotient q and remainder r such that $A = Bq + r$.

Then, note that $gcd(A, B) = gcd(B, r)$.

Repeat the process until $r = 0$.

Example 1

Use the Euclidean Algorithm to hand-calculate the greatest common divisor of each pair of integers.

294 and 60

832 and 10933

It	0	1	2	3
A	10933	832	117	13
B	832	117	13	0
q		13	7	9
r		117	13	0
gcd				13

$10933 = 832(13) + 117$

$832 = 117(7) + 13$

$117 = 13(9) + 0$

GCD 2100, 1320

$q = 1$: $2100 - 1320 = 780$

$= gcd(1320, 780) = gcd(780, 540) = gcd(540, 240) = gcd(60, 240)$

$1320 - 780 = 540$; $780 - 540 = 240$; $540 - 480 = 60$; $240 \div 60 = 4$, remainder 0

$gcd(60, 0) = 60$

Relatively Prime

Two integers are said to be relatively prime iff their greatest common divisor is 1.

2

Section 5.1

Sequences

Sequences

A sequence is a function whose domain is either all the integers between (and including) two given integers or all the integers greater than or equal to a given integer.

Alternately, a sequence is an ordered list of numbers of the form

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

Each individual element a_k is called a term of the sequence.

Explicitly Defined Sequences

Some (but not all) sequences can be defined by giving an explicit formula for the k^{th} term of the sequence.

Example:

The formula

$$a_k = (-1)^k \left(\frac{k}{k+1}\right)$$

defines the sequence

$$-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$$

Example 2

Consider the sequence

$$\frac{2}{3}, \frac{5}{6}, \frac{8}{9}, \frac{11}{12}, \ldots$$

a) Find the next two terms in the sequence.

b) Write an expression for the k^{th} term of the sequence.

Example 3

Consider the sequence

$$1, \frac{3}{2}, \frac{9}{5}, 2, \frac{15}{7}, \frac{9}{4}, \ldots$$

a) Find the next two terms in the sequence.

b) Write an expression for the k^{th} term of the sequence.

Factorials

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

n factorial

Special Case:

$$0! = 1$$

Note that factorials are only defined for nonnegative integers.

Example 4

Simplify each of the following factorial expressions.

$$\frac{4!}{6!}$$

$$\frac{(2n+1)!}{(2n-1)!}$$

Example 5

Consider the sequence

$$1, -x, \frac{x^2}{2}, -\frac{x^3}{6}, \frac{x^4}{24}, -\frac{x^5}{120}, \ldots$$

a) Find the next two terms in the sequence.
b) Write an expression for the n^{th} term of the sequence where n begins at 1.
c) Write an expression for the k^{th} term of the sequence where k begins at 0.

Summation Notation

When adding up the values of a sequence, we can use summation notation to concisely express the sum.

If m and n are integers, $m \leq n$, and $a_m, a_{m+1}, \ldots, a_n$ are real numbers (or real-valued expressions), then

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

k is called the index of summation.
m is called the lower limit of the summation.
n is called the upper limit of the summation.

Sigma

Example 6

Evaluate each summation.

$$\sum_{k=1}^{4} k^2$$

$$\sum_{i=2}^{6} (2i-1)$$

Product Notation

When multiplying the values of a sequence, we can use product notation to concisely express the product.

If m and n are integers, $m \leq n$, and $a_m, a_{m+1}, \ldots, a_n$ are real numbers (or real-valued expressions), then

$$\prod_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

k is called the index.
m is called the lower limit.
n is called the upper limit.

Pi

Example 7

Compute the following product.

$$\prod_{k=2}^{4} \left(1 - \frac{1}{k}\right)$$

Properties of Sums and Products

$$\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

$$\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$$

$$\prod_{k=m}^{n}(a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$$

7

Proving Formulas

Monday, October 10, 2022 2:01 PM

CS1200+Lecture+18+...

Section 5.2

Mathematical Induction I: Proving Formulas

The Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

For every integer $n \geq a$, $P(n)$

is true.

Proof by Induction

To prove a statement of the form

For every integer $n \geq a$, $P(n)$

perform the following two steps:

1. (Basis Step) Show that $P(a)$ is true.
2. (Inductive Step) Show that for every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

1

$1+nx \leq (1+x)^n$ for $x > -1$ and integers $n \geq 2$

Base case: $n=2$

$1+2x$ and $(1+x)^2 = 1+2x+x^2$

Since $x^2 \geq 0$, $1+2x \leq 1+2x+x^2$.

I.H. Assume $1+kx \leq (1+x)^k$ for some integer $k \geq 2$

Goal: Prove $1+(k+1)x \leq (1+x)^{k+1}$

$(1+x)^{k+1} = (1+x)(1+x)^k$

note, $x > -1$, $1+x > 0$

$\geq (1+x)(1+kx)$ by the I.H.

$= 1 + kx + x + kx^2$

$= 1+(k+1)x + kx^2 \geq 1+(k+1)x$ as desired.

By induction, $1+nx \leq (1+x)^n$

Proof by Induction

To perform the inductive step, first suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

- This supposition is called the *inductive hypothesis*.

Then, show that $P(k+1)$ is true.

Example 1

Use induction to prove

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

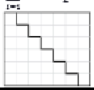
Closed Form

If a sum or product with a variable number of terms is shown to be equal to an expression which does not contain an ellipsis, summation notation, or product notation, we say that the sum is written in closed form.

Basic Summations

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



Basic Summations

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 2

Use a basic summation formula to evaluate each of the following sums.

$$5 + 10 + 15 + 20 + \cdots + 300$$

$$4^3 + 5^3 + \cdots + 10^3$$

Use induction to prove that

$$1 + nx \leq (1+x)^n$$

for real num $x > -1$

and integer $n \geq 2$

Base case: $n = 2$

$$1 + 2x \leq (1+x)^2$$

$$1 + 2x \leq 1 + x^2 + 2x$$

since $x^2 \geq 0$, ✓

I.H. $1 + kx \leq (1+x)^k$ for some int $k \geq 2$

Goal: show $1 + (k+1)x \leq (1+x)^{k+1}$

$$(1+x)^{k+1} = (1+x)(1+x)^k$$

$$\geq (1+x)(1+kx)$$

$$= 1 + kx + x + kx^2$$

$$= 1 + (k+1)x + kx^2$$

$$\geq 1 + (k+1)x$$

$$nCr = \frac{n!}{r!(n-r)!}$$

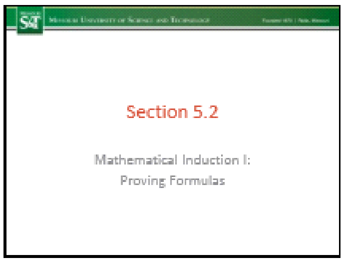
or

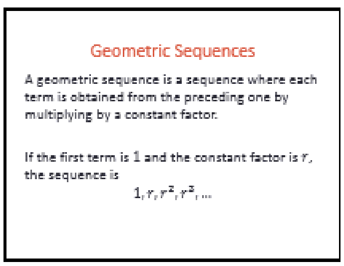
$$\binom{n}{r}$$

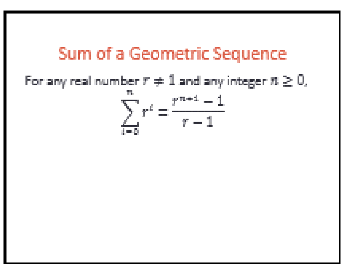
Geometric Sequences

Wednesday, October 12, 2022 2:02 PM

CS1200+Lecture+19+...







1

0^0 in Discrete Mathematics

In discrete mathematics, we typically define
$$0^0 = 1$$

Elsewhere in mathematics (including calculus), we often say 0^0 is indeterminate.

Example 1

Find a closed form expression for the sum
$$5^3 + 5^4 + 5^5 + \cdots + 5^k$$
where $k \geq 3$ is an integer.

Section 5.3

Mathematical Induction II: Applications

2

Example 2

Prove the statement by induction.

$5^n - 1$ is divisible by 4 for every integer $n \geq 0$.

Base case

$n = 0$: $5^0 - 1 = 1 - 1 = 0$, 0 is divisible by 4

I.H. $5^k - 1$ is divisible by 4 for integer $k \geq 0$

$5^k - 1 = 4r$ some integer r.

Show $5^{k+1} - 1 = 4q$ for some integer q.

$5^{k+1} - 1 = 5^k 5 - 1$

$= (4r + 1)5 - 1$

$= 20r + 5 - 1$

$= 20r + 4$

$= 4(5r + 1)$

$5r + 1$ is an integer, call it q.

Thus, $5^{k+1} - 1 = 4q$

By induction $5^n - 1$ is divisible by 4 ∎

Example 3

Prove the statement by induction.

$n^2 < 2^n$ for every integer $n \geq 5$.

Let $n = 5$ $\quad 5^2 < 2^5 - 25 < 32$ ✓

I.H. $k^2 < 2^k$ for integer $k \geq 5$

Goal: Show $(k+1)^2 < 2^{k+1}$

$(k+1)^2 = k^2 + 2k + 1$

$2^n < (n+2)!$ for $n \geq 0$

Proof:

Base case: $2^0 < (0+2)! \rightarrow 1 < 2$ ✓

I.H.: $2^k < (k+2)!$ for some integer $k \geq 0$

Goal: Show $2^{k+1} < (k+1+2)!$

$2^{k+1} = 2^k 2$

$< (k+2)!\, 2$ by I.H.

$< (k+2)!\,(k+3)$ since $k + 3 \geq 3$

$= (k+3)!$ as desired

Thus, by induction, $2^n < (n+2)!$ for $n \geq 0$.

3

Recursion

Friday, October 14, 2022 2:03 PM

CS1200+Lecture+20+...

Sections 5.6 and 5.7

Defining Sequences Recursively

Solving Recurrence Relations by Iteration

Recursively Defined Sequences

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to one or more of its predecessors. The initial conditions for such a relation specify the specific values of a_0 (and, if necessary, the next few values as well).

Example 1

Find the first four terms of each sequence.

a) $a_k = 2a_{k-1} + k$ for every integer $k \geq 2$
$a_1 = 1$

b) $u_k = ku_{k-1} - u_{k-2}$ for every integer $k \geq 3$
$u_1 = 1, u_2 = 1$

$a_1 = 1$ $u_1 = 1$

$a_2 = 4$ $u_2 = 1$

$a_3 = 11$ $u_3 = 2$

$a_4 = 26$ $u_4 = 7$

1

Example 2

A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions:

1. Rabbit pairs are not fertile during the first month of life but thereafter give birth to one new male/female pair at the end of every month.
2. No rabbits die.

How many rabbits will there be at the end of the year?

$r_0 = 1$
$r_1 = 1$
$r_2 = 1 + 1 = 2$
$r_3 = 2 + 1 = 3$
$r_4 = 5$
$r_5 = 8$
$r_6 = 13$
$r_7 = 21$
$r_8 = 34$
$r_9 = 55$
$r_{10} = 89$
$r_{11} = 144$
$r_{12} = 233$ pairs
$r_{13} = 377$
$r_{14} = 610$

The Fibonacci Sequence

$$F_0 = 1$$
$$F_1 = 1$$
$$F_k = F_{k-1} + F_{k-2}$$

Example 3

The Towers of Hanoi puzzle consists of disks with holes in their centers, piled in order of decreasing size on one pole in a row of three. The goal is to move all the disks one by one from one pole to another, never placing a larger disk on top of a smaller one, and ending with the entire pile on a different pole.

How many moves would be required to move a pile of 7 disks?

2

Example 4

Use induction to prove the formula you found in Example 3.

B.C.

$n=1$, $h_1 = 2^1 - 1 = 1$ ✓

I.H. $h_k = 2^k - 1$ for integer $k \geq 1$

Show $h_{k+1} = 2^{k+1} - 1$ using recursive def.

$h_{k+1} = 2h_k + 1$

$= 2(2^k - 1) + 1$ by I.H.

$= 2^{k+1} - 2 + 1$

by induction, $h_n = 2^n - 1$ ∎

Arithmetic Sequences

A sequence $a_0, a_1, a_2, \ldots$ is called an arithmetic sequence iff there is a constant d such that

$$a_k = a_{k-1} + d$$

for each integer $k \geq 1$.

An explicit formula for such a sequence is

$$a_n = a_0 + dn$$

for every integer $n \geq 0$.

Geometric Sequences

A sequence $a_0, a_1, a_2, \ldots$ is called a geometric sequence iff there is a constant r such that

$$a_k = r a_{k-1}$$

for each integer $k \geq 1$.

An explicit formula for such a sequence is

$$a_n = a_0 r^n$$

for every integer $n \geq 0$.

1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255

254

Example 5

Use iteration to find an explicit formula for the sequence.

$b_k = \frac{b_{k-1}}{1+b_{k-1}}$ for each integer $k \geq 1$

$b_0 = 1$

Then, verify the correctness of the formula using induction.

$b_0 = 1$

$b_1 = \frac{1}{2}$

$b_2 = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$

$b_3 = \frac{\frac{1}{3}}{1+\frac{1}{3}} \quad \frac{1}{4}$

$b_n = \frac{1}{1+n}$

4

Set Theory

Friday, October 21, 2022 2:02 PM

CS1200+Lecture+21+...

Section 6.1

Set Theory: Definitions and the Element Method of Proof

Sets

If S is a set and $P(x)$ is a property that elements of S may or may not satisfy, then a set A may be defined by writing

$$A = \{x \in S \mid P(x)\}$$

which is read as "A is the set of all x in S such that $P(x)$ is true."

Subsets

We can formally define a subset as follows:

$$A \subseteq B \Leftrightarrow \forall x, \text{ if } x \in A \text{ then } x \in B.$$

The negation is

$$A \nsubseteq B \Leftrightarrow \exists x \text{ such that } x \in A \text{ and } x \notin B.$$

Proper Subsets

A is called a proper subset of B (denoted $A \subset B$) iff

1) $A \subseteq B$, and
2) there is at least one element in B which is not in A.

Equality of Sets

Given sets A and B, A equals B (written $A = B$) iff every element of A is in B and every element of B is in A.

Symbolically,

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Proving Subsets

Let sets X and Y be given. To prove $X \subseteq Y$,

1) **suppose** that x is a particular but arbitrarily chosen element of X, and then
2) **show** that x is an element of Y.

2

Example 1

Let sets B and C be defined as follows:

$B = \{y \in \mathbb{Z} | y = 10b - 3 \text{ for some integer } b\}$

$C = \{z \in \mathbb{Z} | z = 10c + 7 \text{ for some integer } c\}$

Prove or disprove each of the following statements.

a) $B \subseteq C$

b) $C \subseteq B$

c) $B = C$

Universal Set

A universal set is the set containing all objects or elements and of which all other sets are subsets.

Common examples:

$\mathbb{R}$

$\mathbb{Z}$

$\mathbb{R}^2$ (the set of all two-dimensional real vectors)

Operations on Sets

Let A and B be subsets of a universal set U.

The union of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A and B.

$$A \cup B = \{x \in U | x \in A \text{ or } x \in B\}$$

3

Operations on Sets

Let A and B be subsets of a universal set U.

The intersection of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.

$$A \cap B = \{x \in U | x \in A \text{ and } x \in B\}$$

Operations on Sets

Let A and B be subsets of a universal set U.

The difference of B minus A (or the relative complement of A in B), denoted $B - A$, is the set of all elements that are in B but not A.

$$B - A = \{x \in U | x \in B \text{ and } x \notin A\}$$

Operations on Sets

Let A and B be subsets of a universal set U.

The complement of A, denoted A^c, is the set of all elements in U that are not in A.

$$A^c = \{x \in U | x \notin A\}$$

Interval Notation

Given real numbers $a \leq b$,

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$
$$(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$$
$$[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$$
$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded on either the right or the left.

Example 2

Let the universal set be $\mathbb{R}$, and let

$$A = \{x \in \mathbb{R} | 0 < x \leq 2\}$$
$$B = \{x \in \mathbb{R} | 1 \leq x < 4\}$$
$$C = \{x \in \mathbb{R} | 3 \leq x < 9\}$$

Find each of the following.

$A \cup B$ (0, 4)

$A \cap B$ [1, 2]

$A \cup C$ (0, 2] ∪ [3, 9)

$A \cap C$ ∅

Example 2

Let the universal set be $\mathbb{R}$, and let

$$A = \{x \in \mathbb{R} | 0 < x \leq 2\}$$
$$B = \{x \in \mathbb{R} | 1 \leq x < 4\}$$
$$C = \{x \in \mathbb{R} | 3 \leq x < 9\}$$

Find each of the following.

$B - A$ (2, 4)

A^c (-∞, 0] ∪ (2, ∞)

Operations on Indexed Collections of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one integer } 0 \le i \le n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one integer } i \ge 0\}$$

Operations on Indexed Collections of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for every integer } 0 \le i \le n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for every integer } i \ge 0\}$$

Example 3

Let $D_i = [-2i, 2i]$ for each nonnegative integer i. Find each of the following quantities.

$$\bigcup_{i=0}^{4} D_i = [-8, 8]$$

$$\bigcap_{i=0}^{4} D_i \quad \{0\}$$

The Empty Set

The empty set (or null set) ∅ is the unique set containing no elements.

Disjoint Sets

Two sets are disjoint iff they have no elements in common.

Symbolically,

A and B are disjoint $\Leftrightarrow A \cap B = \emptyset$

Mutually Disjoint Sets

Sets $A_0, A_1, A_2, \ldots$ are mutually disjoint (or pairwise disjoint or nonoverlapping) iff no two sets A_i and A_j with distinct subscripts have any elements in common.

Partition of a Set

A finite or infinite collection of nonempty sets $\{A_0, A_1, A_2, \ldots\}$ is a partition of a set A iff

1) A is the union of all the A_i, and
2) the sets $A_0, A_1, A_2, \ldots$ are mutually disjoint.

Example 4

Consider the set $A = \{a, b, c, d, e, f, g\}$.
Which of the following is a partition of A?

$\{\{a, c, e\}, \{b, d, f\}, \{c, g\}\}$ ✗ c in A_1, A_3

$\{\{a, b\}, \{c, d\}, \{f, g\}\}$ ✗ no e

$\{\{a, c\}, \{b, d, g\}, \{e, f\}\}$ ✓

Power Sets

Given a set A, the power set of A, denoted $\mathcal{P}(a)$, is the set of all subsets of A.

Example 5

Let $A = \{1,2,3,4\}$

Find $\mathcal{P}(a)$.

{{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},....}

9

Properties of Sets

Monday, October 24, 2022 2:00 PM

CS1200+Lecture+22+...

Section 6.2

Properties of Sets

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1) $x \in X \cup Y$ iff $x \in X$ or $x \in Y$.
2) $x \in X \cap Y$ iff $x \in X$ and $x \in Y$.
3) $x \in X - Y$ iff $x \in X$ and $x \notin Y$.
4) $x \in X^c$ iff $x \notin X$.
5) $(x, y) \in X \times Y$ iff $x \in X$ and $y \in Y$.

Subset Relations and Order of Operations

The operations of union, intersection, and difference take precedence over set inclusion

1

Some Subset Relations

For all sets A and B,
$A \cap B \subseteq A$ and $A \cap B \subseteq B$.

For all sets A and B,
$A \subseteq A \cup B$ and $B \subseteq A \cup B$.

For all sets A, B, and C,
if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Example 1

Prove the following statement:

For all sets A, B, and C,
if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Let $A \subseteq B$ and $B \subseteq C$
let $x \subseteq A$
since $A \subseteq B$, $x \in B$
since $B \subseteq C$ $x \in C$
since $x \in C$, $A \subseteq C$ ■

Set Identities

Let all sets referenced below be subsets of a universal set U.

Set Identities

Let all sets referenced below be subsets of a universal set U.

Commutative Laws:

For all sets A and B,

$$A \cup B = B \cup A$$

and

$$A \cap B = B \cap A$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Associative Laws:

For all sets A, B, and C,

$$(A \cup B) \cup C = A \cup (B \cup C)$$

and

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Distributive Laws:

For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 2

Prove the distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

for all sets A, B, and C.

P1: Let A, B, C be sets. Prove $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

goal: show $x \in (A \cup B) \cap (A \cup C)$

Case 1: $x \in A$

If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$

since $x \in A \cup B$ and $x \in A \cup C$, $x \in (A \cup B) \cap (A \cup C)$

Case 2: $x \in B \cap C$

if $x \in B \cap C$, then $x \in B$ and $x \in C$

since $x \in B$, $x \in A \cup B$

since $x \in C$, $x \in A \cup C$

thus, $x \in (A \cup B) \cap (A \cup C)$

Hence, in both cases, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

P2: prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

...

Set Identities

Let all sets referenced below be subsets of a universal set U.

Identity Laws:

For every set A,

$$A \cup \emptyset = A$$

and

$$A \cap U = A$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Complement Laws:

For every set A,

$$A \cup A^c = U$$

and

$$A \cap A^c = \emptyset$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Double Complement Law:
For every set A,
$$(A^c)^c = A$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Idempotent Laws:
For every set A,
$$A \cup A = A$$
and
$$A \cap A = A$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Universal Bound Laws:
For every set A,
$$A \cup U = U$$
and
$$A \cap \emptyset = \emptyset$$

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Set Identities

Let all sets referenced below be subsets of a universal set U.

De Morgan's Laws:

For all sets A and B,

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Absorption Laws:

For all sets A and B,

$$A \cup (A \cap B) = A$$

and

$$A \cap (A \cup B) = A$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Complements of U and $\emptyset$:

$$U^c = \emptyset$$

and

$$\emptyset^c = U$$

Set Identities

Let all sets referenced below be subsets of a universal set U.

Set Difference Law:

For all sets A and B,

$$A - B = A \cap B^c$$

Theorem

For any sets A and B, if $A \subseteq B$, then

1) $A \cap B = A$, and
2) $A \cup B = B$.

Example 3

Prove the following theorem:

If E is a set with no elements and A is any set, then $E \subseteq A$.

Suppose this isn't true.
That is, E with no elements,
such that $E \not\subseteq A$
Then, there is an element of E
which is not in A
Since E contains no elements,
This is a contradiction.
Thus, $E \subseteq A$ ■

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Example 4

Prove the following theorem:

There is only one set with no elements

Assume two distinct sets E_1, E_2.
Both have no elements
By the preceding theorem,
$E_1 \subseteq E_2$ and $E_2 \subseteq E_1$
Therefore $E_1 = E_2$ ■

Proving a Set is Empty

To prove that a set is empty, suppose the set has an element and derive a contradiction.

Example 5

Prove the following theorem:

For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

Let $x \in A \cap B^c$.
Then $x \in A$ and $x \in B^c$.
Hence $x \in A$ and $x \notin B$
However, $A \subseteq B$.
This is a contradiction.
Thus, $A \cap B^c = \emptyset$

8

Functions

Monday, October 31, 2022 2:00 PM

CS1200+Lecture+23+...

Section 7.1

Functions Defined on General Sets

Functions (more formally)

A function f from a set X to a set Y, denoted $f: X \to Y$, is a relation from X (the domain of f) to Y (the codomain of f) which satisfies two properties:

1) Every element in X is related to some element in Y, and
2) No element in X is related to more than one element in Y.

Functions (more formally)

The unique element to which f sends an element x in X is called $f(x)$ and can be referred to in words as:

- f of x
- The output of f for the input x
- The value of f at x
- The image of x under f

1

Functions (more formally)

The set of all values of f is called the range of f or the image of X under f.

Symbolically,
range of $f = \{y \in Y | y = f(x)$ for some $x \in X\}$

Functions (more formally)

Given an element y in Y, there may exist element in X with y as their image. When x is an element such that $f(x) = y$, then x is called a preimage of y or an inverse image of y. The set of all inverse images of y is called the inverse image of y

Symbolically,
inverse image of $y = \{x \in X | f(x) = y\}$

Equality of Functions

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ iff $F(x) = G(x)$ for every $x \in X$.

The Identity Function

Given a set X, define a function I_X from X to X by

$$I_X(x) = x$$

for each $x \in X$.

I_X is called the identity function on X

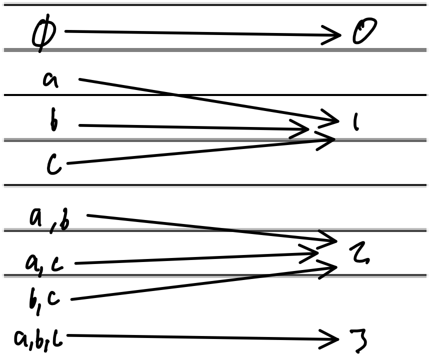
Example 1

Recall that $\mathcal{P}(A)$ denotes the set of all subsets of A.

Define a function $F: \mathcal{P}(\{a, b, c\}) \to \mathbb{Z}^{nonneg}$ as follows:

$F(X) =$ the number of elements in X

Draw an arrow diagram for F.



Example 2

Let $J_5 = \{0,1,2,3,4\}$ and define a function $G: J_5 \times J_5 \to J_5 \times J_5$ as follows:
For each $(a, b) \in J_5 \times J_5$,
$G(a, b) = \big((2a + 1)\ mod\ 5, (3b - 2)\ mod\ 5\big)$

Find the following.

$G(3,4)$
$G(1,0)$

$((2(3)+1) \bmod 5, (3(4)-2) \bmod 5) = (2, 0)$

$((2(1)+1) \bmod 5, (3(0)-2) \bmod 5) = (3, 3)$

$-2 \bmod 5 = 3 \bmod 5$

3

Logarithms

Let b be a positive real number with $b \neq 1$. For each positive real number x, the logarithm with base b of x is the exponent to which b must be raised to obtain x.

Symbolically,

$$\log_b x = y \Leftrightarrow b^y = x$$

The logarithmic function with base b is the function from $\mathbb{R}^+$ to $\mathbb{R}$ that takes each positive real number x to $\log_b x$.

Example 3

Find each of the following values.

$\log_4 16$ 2

$\log_4 \frac{1}{2}$ $-\frac{1}{2}$

$\log_4 4^m$ m

$4^{\log_4 m}$ m

Example 4

Let S_n be the set of all strings of 0's and 1's of length n. The Hamming distance function $H: S_n \times S_n \to \mathbb{Z}^{nonneg}$ is defined as follows: For each pair of strings $(s, t) \in S_n \times S_n$, $H(s, t) =$ the number of positions in which s and t have different values.

Find $H(00101, 01110)$.

Boolean Functions

An n-place Boolean function f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0,1\}$.

More formally, the domain is the Cartesian product of n copies of $\{0,1\}$, denoted $\{0,1\}^n$.

Example 5

Consider the three-place Boolean function f defined by the following rule:

For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (4x_1 + 3x_2 + 2x_3)\ mod\ 2$$

Find $f(1,1,1)$ and $f(0,0,1)$.

(4+3+2) a mod 2 = 1

(0+0+2) mod 2 = 0

Functions Acting on Sets

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y | y = f(x) \text{ for some } x \in A\}$$

and

$$f^{-1}(C) = \{x \in X | f(x) \in C\}.$$

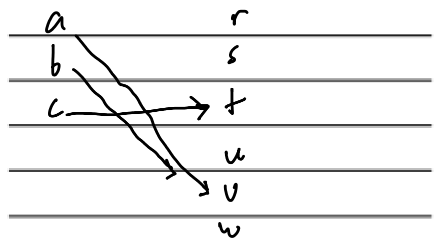
$f(A)$ is called the image of A.
$f^{-1}(C)$ is called the inverse image of C.

Example 6

Let $X = \{a, b, c\}$ and $Y = \{r, s, t, u, v, w\}$.
Define $f: X \to Y$ as follows:

$$f(a) = v, f(b) = v, f(c) = t$$

1) Draw an arrow diagram for f.
2) Let $A = \{a, b\}$, $C = \{t\}$, $D = \{u, v\}$, and $E = \{r, s\}$. Find $f(A)$, $f(X)$, $f^{-1}(C)$, $f^{-1}(D)$, $f^{-1}(E)$, and $f^{-1}(Y)$.



$f(A) = \{v\}$

$f(X) = \{t, v\}$

$f^{-1}(C) = \{c\}$

$f^{-1}(D) = \{a, b\}$

$f^{-1}(E) = \emptyset$

$f^{-1}(Y) = \{a, b, c\}$

6

Inverse Functions

Wednesday, November 2, 2022 2:00 PM

CS1200+Lecture+24+...

Section 7.2

One-to-One, Onto, and Inverse Functions

Recall: Functions (more formally)

A function f from a set X to a set Y, denoted $f: X \to Y$, is a relation from X (the domain of f) to Y (the codomain of f) which satisfies two properties:

1) Every element in X is related to some element in Y, and
2) No element in X is related to more than one element in Y.

One-to-One Functions

Let $f: X \to Y$ be a function.

f is one-to-one (or injective) iff for all elements x_1 and x_2 in X,

if $f(x_1) = f(x_2)$ then $x_1 = x_2$

or, equivalently,

if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

1

Example 1

Define functions $F: X \to Y$ and $G: X \to Y$ by the arrow diagrams below.

Is F one-to-one? No

Is G one-to-one? No

One-to-One Functions on Infinite Sets

To prove a function f is one-to-one, use direct proof as follows:

Suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$.

Show that $x_1 = x_2$.

One-to-One Functions on Infinite Sets

To prove a function f is <u>not</u> one-to-one, we typically find elements x_1 and x_2 in X such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

$m = 2 - 3m$

$+3m$

$4m = 2$

$m = \frac{1}{2}$

$\frac{1}{2} = 2 - \frac{3}{2}$

$\frac{3x_1 - 1}{x_1} = \frac{3x_2 - 1}{x_2}$

$- \frac{1}{x_1} = - \frac{1}{x_2}$

$x_1 = x_2$

Example 2

Define $g: \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 8n + 3$ for each integer n.

Is g one-to-one? Prove or give a counterexample.

Suppose $a, b \in \mathbb{Z}$ such that $g(a) = g(b)$.

$8a + 3 = 8b + 3$

$\frac{8a}{8} = \frac{8b}{8}$

$a = b$

therefore g is one-to-one

Example 3

Define $H: \mathbb{R} \to \mathbb{R}$ by the rule $H(x) = x^2$ for each real number x.

Is H one-to-one? Prove or give a counterexample.

$H(2) = 2^2 = 4$

$H(-2) = (-2)^2 = 4$

Not one-to-one

Onto Functions

Let $f: X \to Y$ be a function.

f is onto (or surjective) iff given any element y in Y, it is possible to find an element x in X with the property that $y = f(x)$.

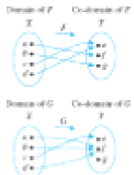
f is not onto iff there exists an element y in Y for which it is not possible to find an element x in X with the property that $y = f(x)$.

Example 1 (continued)

Define functions $F: X \to Y$ and $G: X \to Y$ by the arrow diagrams below.

Is F onto? Yes

Is G onto? No



Onto Functions on Infinite Sets

To prove a function f is one-to-one, generalize from the generic particular:

Suppose y is any element of Y.

Show that there is an element x in X with $f(x) = y$.

Onto Functions on Infinite Sets

To prove a function f is not onto, we typically find an element y of Y such that $y \neq f(x)$ for any x in X.

Example 2 (continued)

Define $g: \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 8n + 3$ for each integer n.

Is g onto? Prove or give a counterexample.

$g(n) = 8n + 3$

$10 = 8n + 3$

$7 = 8n$

$\frac{7}{8} = n$ not an intger, not onto.

Example 3 (continued)

Define $H: \mathbb{R} \to \mathbb{R}$ by the rule $H(x) = x^2$ for each real number x.

Is H onto? Prove or give a counterexample.

$x^2 = 4$

$x = \pm 2i$, not real,

H is not onto.

Exponential Function with base b

The exponential function with base b is the function from $\mathbb{R}$ to $\mathbb{R}^+$ defined as

$$\exp_b x = b^x$$

for all real numbers x, where

$$b^0 = 1$$

and

$$b^{-x} = \frac{1}{b^x}$$

5

Laws of Exponents

If a and b are both positive and x and y are both real, then

$$a^{x+y} = a^x a^y$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

Properties of Logarithms

For positive real numbers $b \neq 1$, x, and y and for every real number a, we have

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^a = a \log_b x$$

Change of Base Formula

For any positive bases $a \neq 1$ and $b \neq 1$,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example 4

Evaluate $\log_2 7$ using a calculator.

$\log_2 7 = \frac{\log_e 7}{\log_e 2} = \frac{\ln 7}{\ln 2}$

$\log_2 7 = \frac{\log_{10} 7}{\log_{10} 2}$

One-to-One Correspondences

A one-to-one correspondence (or bijection) from a set X to a set Y is a function $f: X \to Y$ that is both one-to-one and onto.

Example 2 (continued)

Define $g: \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 8n + 3$ for each integer n.

Is g a one-to-one correspondence?

No. not one-to-one and onto

Inverse Functions

Suppose $f: X \to Y$ is both one-to-one and onto (i.e. a one-to-one correspondence). Then, there exists a function $f^{-1}: Y \to X$ such that $f^{-1}(y)$ is the unique element x in X such that $f(x) = y$.

Symbolically,

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Example 2 (continued)

Define $g: \mathbb{Z} \to \mathbb{Z}$ by the rule $g(n) = 8n + 3$ for each integer n.

Find its inverse function g^{-1}.

$\mathbb{R} \to \mathbb{R}$

$h(x) = 8x + 3$

$y = 8x + 3$

$x = 8y + 3$

$y = \frac{x-3}{8}$

$f^{-1}(x) = \frac{x-3}{8}$

Theorem

If X and Y are sets and $f: X \to Y$ is one-to-one and onto, then $f^{-1}: Y \to X$ is also one-to-one and onto.

$x = 2 - 5y$

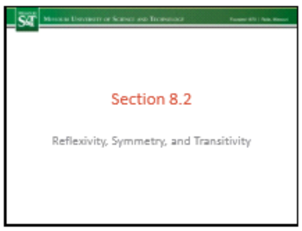
$\frac{x-2}{-5} = y$

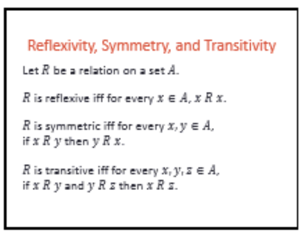
8

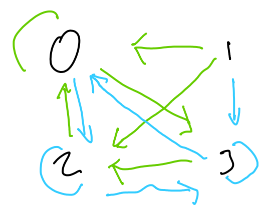
Reflexivity, Symmetry, Transitivity

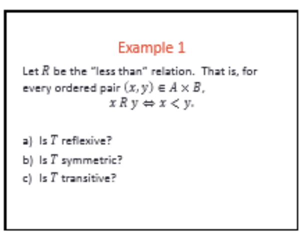
Wednesday, November 9, 2022 2:03 PM

CS1200+Lecture+26+...









No, $x \not< x$

No, $x < y$ and $y < x$

Yes, $x < y$ and $y < z$, then $x < z$

1

Example 2

Recall that the congruence modulo 3 relation, T, is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows:
For all integers m and n,

$$m\ T\ n \Leftrightarrow 3|(m-n).$$

a) Is T reflexive?
b) Is T symmetric?
c) Is T transitive?

Yes, 3|(n−n)

Yes. long proof

3|(q−r) q−r = 3j
3|(p−q) p−q = 3k
(p−q)+(q−r) = 3(j+k)
Yes.

The Transitive Closure of a Relation

Let R be a relation on a set A. The transitive closure of R is the relation R^t on A which satisfies the following three properties:

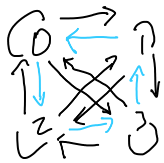
1) R^t is transitive.
2) $R \subseteq R^t$.
3) If S is any other transitive relation that contains R, then $R^t \subseteq S$.

The Transitive Closure of a Relation

In other words, the transitive closure of a relation is the smallest transitive relation that contains the relation.

Example 3

Let $T = \{(0,2),(1,0),(2,3),(3,1)\}$ be a relation defined on $A = \{0,1,2,3\}$. Find T^+, the transitive closure of T.



need to add

(1,2),
(0,3),
(1,3),
(3,0),
(3,3),
(0,0),
(1,1),
(0,1),
(2,1)
(2,0)
(2,2)
(3,2)

Relations on Sets

Wednesday, November 9, 2022 2:03 PM

CS1200+Lecture+25+...

Section 8.1

Relations on Sets

Binary Relations

A binary relation is a subset of a Cartesian product of two sets.

Example 1

The congruence modulo 3 relation, T, is defined from $\mathbb{Z}$ to $\mathbb{Z}$ as follows: For all integers m and n,

$$m\,T\,n \Leftrightarrow 3 \mid (m-n).$$

a) Is $10\,T\,1$?
   Is $1\,T\,10$?
   Is $(2,2) \in T$?
   Is $(8,1) \in T$?

b) List five integers n such that $n\,T\,1$

It can be shown that $m\,T\,n$ iff $m \bmod 3 = n \bmod 3$.

$10\,T\,1$: $\frac{10-1}{3}$ ✓

$1\,T\,10$. $\frac{1-10}{3}$ ✓

$(2,2) \in T$: $\frac{2-2}{3}$ ✓

$(8,1) \in T$: $\frac{8-1}{3}$ ✗

1

Example 2

Let $X = \{a, b, c\}$ and recall that $\mathcal{P}(X)$ is the power set of X. Define a relation S on $\mathcal{P}(X)$ as follows. For all sets A and B in $\mathcal{P}(X)$,
$A\ S\ B \Leftrightarrow A$ has the same number of elements as B.

a) Is $\{a, b\}\ S\ \{b, c\}$?
b) Is $\{a\}\ S\ \{a, b\}$?

The Inverse of a Relation

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

In other words, for all $x \in A$ and $y \in B$,

$$(y, x) \in R^{-1} \Leftrightarrow (x, y) \in R.$$

Example 3

Let $A = \{3,4,5\}$ and $B = \{4,5,6\}$ and let R be the "less than" relation. That is, for every ordered pair $(x, y) \in A \times B$,

$$x\ R\ y \Leftrightarrow x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

R	R^{-1}
3,4	4,3
3,5	5,3
3,6	6,3
4,5	5,4
4,6	6,4
5,6	6,5

2

Relations on a Set

A relation on a set A is a relation from A to A.

The Directed Graph of a Relation

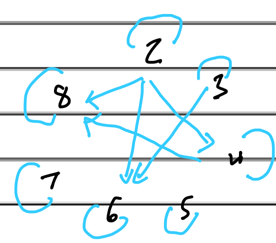
For all points x and y in A, there is an arrow from x to y iff $x\ R\ y$ or, equivalently, $(x, y) \in R$.

Example 4

Let $A = \{2,3,4,5,6,7,8\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x\ R\ y \Leftrightarrow x | y$$

Draw the directed graph of the relation.

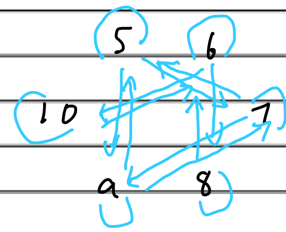


Example 5

Let $A = \{5,6,7,8,9,10\}$ and define a relation S on A as follows: For every $x, y \in A$,

$$x\ S\ y \Leftrightarrow 2 | (x - y)$$

Draw the directed graph of the relation.



Unions and Intersections of Relations

Given two relations R and S from A to B,

$R \cup S = \{(x, y) \in A \times B | (x, y) \in R \text{ or } (x, y) \in S\}$

$R \cap S = \{(x, y) \in A \times B | (x, y) \in R \text{ and } (x, y) \in S\}$

Example 6

Let $A = \{2,4\}$ and $B = \{6,8,10\}$ and define relations R and S from A to B as follows: For every $(x, y) \in A \times B$,

$$x\ R\ y \Leftrightarrow x | y$$

and

$$x\ S\ y \Leftrightarrow y = 4 - x.$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$, and $R \cap S$.

R — S

2, 6 — ∅

2, 8

2, 10

(4, 8)

Example 7

Define relations R and S on $\mathbb{R}$ as follows:

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = |x|\}$$
$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | y = 1\}$$

Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

Equivalence Relations

Friday, November 11, 2022 2:03 PM

CS1200+Lecture+27+...

Section 8.3

Equivalence Relations

Recall: Partition of a Set

A finite or infinite collection $\{A_0, A_1, A_2, \ldots\}$ of nonempty subsets of a set A is a partition of A iff

1) A is the union of all the A_i, and
2) the sets $A_0, A_1, A_2, \ldots$ are mutually disjoint.

The Relation Induced by a Partition

Given a partition of a set A, the relation induced by the partition, R, is defined on A as follows:

For every $x, y \in A$,

$x \, R \, y \Leftrightarrow$ there is a subset A_i of the partition such that both x and y are in A_i

1

0: -6, -3, 0, 3

1: -5, -4, -1, 1, 4

2: -5, -2, -1, 1, 2

3: -6, -3, 0, 3 3-36

4. -5, -4, -1, 1, 4

-5: -5, -4, -2, 2, 4 25-1

25-1

25-16

25-4

Example 1

The partition $\{\{0\},\{1,3,4\},\{2\}\}$ induces a relation R on $\{0,1,2,3,4\}$. Find the ordered pairs in R.

(0,0) (2,2) (1,1) (3,3)
(1,3) (3,1) (1,4) (4,1)
(4,4) (3,4) (4,3)

Equivalence Relations

Let A be a set and R be a relation on A. R is an equivalence relation iff R is reflexive, symmetric, and transitive.

Example 2

Let A be the set of all S&T undergraduate students, and let R be the relation defined on A as follows:
For every $x, y \in A$,

$$x\ R\ y \Leftrightarrow x \text{ has the same major as } y.$$

Prove that the relation is an equivalence relation.

Equivalence Classes

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the equivalence class of a, denoted $[a]$ and sometimes just called the class of a, is the set of all elements x in A such that x is related to a by R.

Symbolically,

$$[a] = \{x \in A \mid x\ R\ a\}$$

Example 2 (continued)

Let A be the set of all S&T undergraduate students, and let R be the relation defined on A as follows:
For every $x, y \in A$,

$$x\ R\ y \Leftrightarrow x \text{ has the same major as } y.$$

Describe the distinct equivalence classes of the relation. (Assume "undeclared" is a major.)

Example 3

Let $A = \{ \quad -4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and define an equivalence relation R on A as follows:
For every $(m, n) \in A$,

$$m\ R\ n \Leftrightarrow 4 \mid (m^2 - n^2)$$

Find the distinct equivalence classes of R.

3

Representatives of Equivalence Classes

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A representative of the class S is any element a such that $[a] = S$.

Lemma

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A.
If $a\ R\ b$, then $[a] = [b]$.

Another Lemma

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A.
Then, either

$$[a] \cap [b] = \emptyset$$

or

$$[a] = [b].$$

4

The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

Congruence $mod\ d$

Let m and n be integers and let d be a positive integer. We say that m is congruent to n modulo d and write

$$m \equiv n \ (mod\ d)$$

iff

$$d \mid (m - n).$$

Example 4

Determine which of the following congruence relations are true and which are false.

$$17 \equiv 2 \ (mod\ 5)$$
$$3 \equiv -1 \ (mod\ 4)$$
$$-11 \equiv -5 \ (mod\ 3)$$
$$6 \equiv 13 \ (mod\ 2)$$

5

Example 5

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero; that is, $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Define a relation R on A as follows:

For all pairs (a, b) and (c, d) in A,

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc.$$

a) Prove that R is transitive.

Let $(a,b) R (c,d)$ and $(c,d) R (e,f)$

Goal: show $(a,b) R (e,f)$

$ad = bc$ and $cf = de$

$adf = bcf$ and $bcf = bde$

$adf = bde$

$af = be$ $\quad d \neq 0$

$(a,b) R (e,f)$

Example 5

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero; that is, $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$.

Define a relation R on A as follows:

For all pairs (a, b) and (c, d) in A,

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc.$$

b) Describe the distinct equivalence classes of R.

Rational Numbers

As we have (at least partially) seen in the previous example, rational numbers can be defined as equivalence classes of ordered pairs of integers.

Probability

Monday, November 14, 2022 2:00 PM

CS1200+Lecture+28+...

Section 9.1

An Introduction to Probability

Random Processes

When we say that a process is random, it means that when the process takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which specific outcome will occur.

Examples:

Flipping a coin

Rolling a die

Sample Spaces and Events

A sample space is the set of all possible outcomes of a random process or experiment.

An event is a subset of a sample space.

Probability of Equally Likely Events

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the probability of E, denoted $P(E)$, is

$$P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes in } S}$$

Number of Elements in a Set

For any finite set A, $N(A)$ denotes the number of elements in A.

Thus,

$$P(E) = \frac{N(E)}{N(S)}$$

Standard Deck of Cards

A standard deck of cards contains 52 cards divided into four suits. The red suits are diamonds and hearts. The black suits are clubs and spades. Each suit contains 13 cards of the following denominations:

A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king)

The cards J, Q, and K are called face cards.

Example 1

Consider a standard deck of cards and imagine that the deck is so thoroughly shuffled that if you select a card at random, each card is equally likely to be selected. Write each event as a set and compute its probability.

a) The event that the chosen card is red and not a face card.

b) The event that the chosen card is black and has an even number on it.

Example 2

Suppose that each child born is equally likely to be a boy or a girl. Consider a family with exactly three children. Let *BBG* indicate that the first two children born are boys and the last child born is a girl.

1. List the eight elements in the sample space whose outcomes are all possible genders of the three children.
2. Write each of these events as a set and find its probability.
   a) The event that exactly one child is a girl.
   b) The event that at least two children are girls.
   c) The event that no child is a girl.

Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 3

If the largest of 56 consecutive integers is 279, what is the smallest?

Example 4

How many positive two-digit integers are multiples of 3?

What is the probability that a randomly chosen positive two-digit integer is a multiple of 3?

$12 = 3(4)$

$99 = 3(33)$

$33 - 4 + 1 = 30$

$\frac{30}{90} = \frac{1}{3}$

Example 5

There are three doors on the set for a game show – let's call them *A*, *B*, and *C*. If you pick the correct door, you win the prize. You pick door *A*. The host of the show then opens one of the other doors and reveals that there is no prize behind it. Keeping the remaining two doors closed, he asks whether you want to switch your choice to the other closed door or stay with your original choice of door *A*. What should you do if you want to maximize your chance of winning the prize – stay or switch? Or, would the likelihood of winning be the same either way?

A	B	C
P	O	O
O	P	O
O	O	P

Switch	Stay
L	W
W	L
W	L

$\frac{2}{3}$ for switch

Possibly Trees and Multiplication Rule

Wednesday, November 16, 2022 2:01 PM

CS1200+Lecture+29+...

Section 9.2

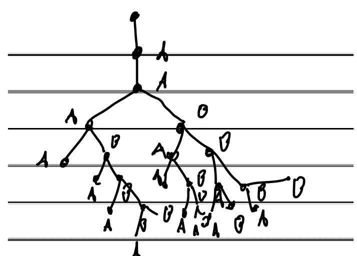
Possibility Trees and the Multiplication Rule

Example 1

In baseball's World Series, the first team to win four games wins the series. Suppose Team A is playing Team B.

a) If Team A wins the first two games, how many ways can the World Series be completed? (Draw a tree.)

b) How many ways can a World Series be played if Team A wins four games in a row?



15 ways

4 ways

The Multiplication Rule

If an operation consists of k steps and

- the first step can be performed in n_1 ways
- the second step can be performed in n_2 ways (regardless of how the first step was performed)

⋮

- the kth step can be performed in n_k ways (regardless of how the preceding steps were performed)

then the entire operation can be performed in

$$n_1 n_2 \cdots n_k$$

ways.

1

Example 2

Hexadecimal numbers are made using the sixteen hexadecimal digits

0,1,2,3,4,5,6,7,8,9, A, B, C, D, E, F.

For example, 9A2D is a hexadecimal number.

How many hexadecimal numbers begin with one of the digits 3 through B, end with one of the digits 5 through F, and are 5 digits long?

9(16)(16)(16)(11)

405, 504

Example 3

How many integers are there from 100 through 999?

How many integers from 100 to 999 have distinct digits?

What is the probability that a randomly chosen three-digit integer has distinct digits?

How many odd integers are there from 100 through 999?

How many odd integers from 100 to 999 have distinct digits?

999 − 100 + 1 = 900 or

choose digits 9(10)(10) = 900

9(9)(8) = $\frac{648}{900}$

8(10)(5) = 400

(5) last digit

(9) middle

7 first

9 9 8 7

5 9 8 4

Permutations

A permutation of a set of objects is an ordering of the objects. For example, the set of elements a, b, and c has six permutations:

$abc, acb, bac, bca, cab, cba$

For any integer $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 4

Six people attend the theatre together and sit in a row with exactly six seats.

a) How many ways can they be seated together in the row?

b) Suppose one of the six is a doctor who must sit on the aisle in case she is paged. How many ways can the people be seated together in the row with the doctor in an aisle seat, assuming there is an aisle at only one end of the row?

c) Repeat part (b) assuming that there is an aisle at both ends of the row.

6! = 720

5!

(2)(5!)

Example 4

Six people attend the theatre together and sit in a row with exactly six seats.

d) Suppose the six people consist of three married couples and each couple wants to sit together with the older partner on the left. How many ways can the six be seated together in the row?

3(2)(1) = 6

r-permutations

An r-permutation of a set of n elements is an ordered selection of r elements taken from the set of n elements.

The number of r-permutations of a set of n elements is

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!}$$

3

Example 5

How many 3-permutations are there of a set of 8 objects?

$$\frac{8!}{(8-3)!}$$

$8(7)(6) = 336$

Example 6

Five people are to be seated around a circular table. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

$\frac{5!}{5} = 4! = 24$

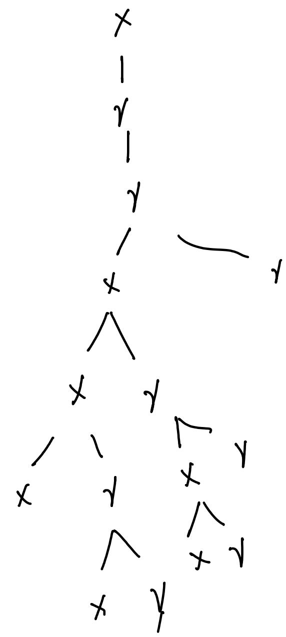
Example 7

In a six-cylinder engine, the even-numbered cylinders are on the left and the odd-numbered cylinders are on the right. A good firing order is a permutation of the numbers 1 to 6 in which right and left sides are alternated. How many possible good firing orders are there which start with a left cylinder?

L R L R L R

3 3 2 2 1 1

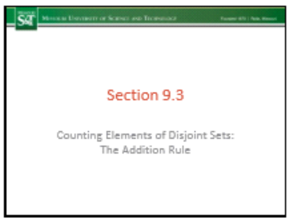
$= 36$

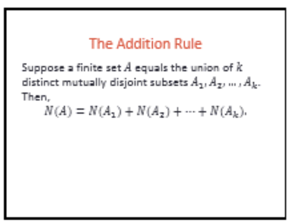


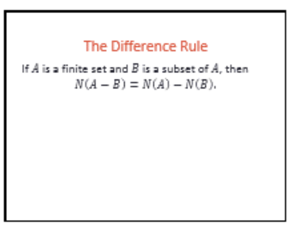
Addition Rule and Pigeonhole Principle

Monday, November 28, 2022 2:03 PM

CS1200+Lecture+30+...







1

Example 1

In Python, identifiers must start with one of 53 symbols: upper- or lower-case letters in the Roman alphabet or an underscore. The initial character may stand alone, or it may be followed by any number of additional characters from a set of 63 symbols (the 53 above plus the ten digits). Certain keywords are set aside and may not be used – in one implementation, there are 31 such reserved keywords, none of which has more than eight characters. How many Python identifiers are there that are less than or equal to eight characters in length?

$53 + 53(63) + 53(63)^2 \ldots + 53(63)^6 + 53(63)^7 - 31$

$= 212,133,167,002,849$

The Probability of the Complement

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

The Inclusion/Exclusion Rule

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

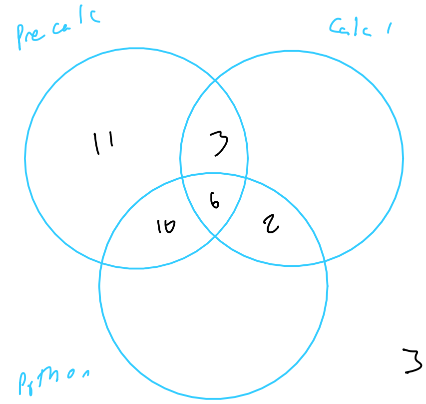
and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Example 2

A discrete math professor passes out a form asking students to check all the math and comp sci courses they have taken. Out of a total of 50 students in the class, she found that

- 30 took precalc
- 18 took Calculus I
- 26 took Python
- 9 took both precalc and Calc I
- 16 took both precalc and Python
- 8 took both Calc I and Python
- 47 took at least one of the three courses



Example 2

a) How many students did not take any of the three courses?
b) How many students took all three courses?
c) How many students took precalc and Calc I but not Python?
d) How many students took precalc but neither Calc I nor Python?

a. 3

b. 6

c. 3

d. 11

$N(P \cup C \cup Y)$

$= N(P) + N(C) + N(Y) - N(P \cap C) - N(P \cap Y) - N(C \cap Y) + N(P \cap C \cap Y)$

$47 = 30 + 18 + 26 - 9 - 16 - 8 + N(P \cap C \cap Y)$

$N(P \cap C \cap Y) = 6$

Section 9.4

The Pigeonhole Principle

The Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one. There must be at least two elements in the domain that have the same image in the co-domain.

Example 3

A small town has only 500 residents. Must there be 2 residents who share the same birthday? Why?

Yes 500 > 366

Example 4

Assuming that all years have 365 days and all birthdays occur with equal probability, how large must n be so that in any randomly chosen group of n people the probability that two or more have the same birthday is at least $\frac{1}{2}$?

23

4

Example 5

Given any set of four integers, must there be two that have the same remainder when divided by 3? Why?

3 remainder possibilities < 4 integers

Yes

7 6 5 4 3 2 1

Example 6

How many cards must you pick from a standard deck of cards to be sure of getting at least one red card?

27

5

Trails, Paths, Circuits

Monday, December 5, 2022 2:02 PM

CS1200+Lecture+31+...

Section 10.1

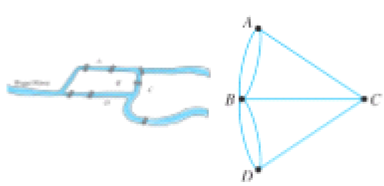
Trails, Paths, and Circuits

The Königsberg Bridge Problem

The town of Königsberg in Prussia was built at a point where two branches of the Pregel River came together. It consisted of an island and some land along the river banks connected with seven bridges (as shown on the next slide). In 1736, Leonhard Euler published a paper answering the following question:

Is it possible for a person to take a walk around town, starting and ending at the same location, and crossing each of the seven bridges exactly once?

The Königsberg Bridge Problem



1

The Königsberg Bridge Problem

Is it possible to find a route through the graph that starts and ends at some vertex and traverses each edge exactly once?

Walks

Let G be a graph and let v and w be vertices in G.

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus, a walk has the form

$$v_0 e_1 v_1 e_2 v_2 e_3 \cdots v_{n-1} e_n v_n$$

where the v_i's represent vertices, $v_0 = v$, $v_n = w$, and the e_i's represent edges. Further, note that v_{i-1} and v_i are the endpoints of e_i.

The trivial walk from v to v consists of the single vertex v.

Trails

A trail from v to w is a walk from v to w that does not contain a repeated edge.

2

Paths

A path from v to w is a trail that does not contain a repeated vertex.

Closed Walks

A closed walk is a walk that starts and ends at the same vertex.

Circuits

A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.

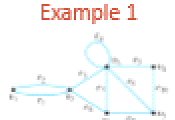
Simple Circuits

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

Summary of Definitions

	Repeated Edge?	Repeated Vertex?	Starts and Ends at the Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

Example 1



Using the graph above, determine whether each walk is a trail, path, circuit, or simple circuit. trail

a) $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$

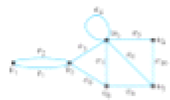
b) $e_1 e_3 e_5 e_5 e_6$ walk

c) $v_2 v_3 v_4 v_5 v_3 v_6 v_2$ circuit

4

4.4

Example 1



Using the graph above, determine whether each walk is a trail, path, circuit, or simple circuit.

d) $v_2 v_3 v_4 v_5 v_6 v_2$ simple circuit

e) $v_1 e_1 v_2 e_1 v_1$ closed walk

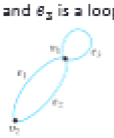
f) v_1 trail

Subgraphs

A graph H is said to be a subgraph of a graph G iff every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

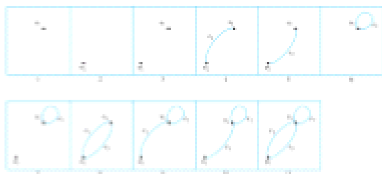
Example 2

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.



Example 2

The 11 subgraphs of G are shown below.



Connectedness

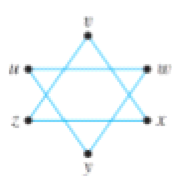
Two vertices v and w of a graph G are connected iff there is a walk from v to w.

A graph G is connected iff given any two vertices v and w in G there is a walk from v to w.

Thus, a graph is connected iff it is possible to travel from any vertex to any other vertex along a sequence of adjacent edges of the graph.

Example 3

Is the following graph connected? no



A Lemma about Connectedness

Let G be a graph.

If vertices v and w are part of a circuit in G and one edge is removed from the circuit, then there still exists a trail from v to w in G.

If G is connected and G contains a circuit, then an edge of the circuit can be removed without disconnecting G.

Connected Components

A graph H is a connected component of a graph G iff

i. H is a subgraph of G;

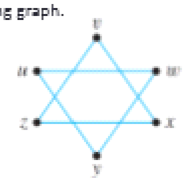
ii. H is connected; and

iii. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

In other words, a connected component is a connected subgraph of largest possible size.

Example 3 (continued)

Find the number of connected components for the following graph.



Euler(ian) Circuits

Let G be a graph. An Euler (or Eulerian) circuit of G is a circuit that contains every vertex and every edge of G.

In other words, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

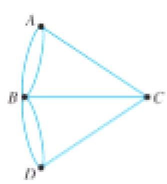
Theorem

A graph G has an Euler circuit iff G is connected and every vertex of G has positive even degree.

The Königsberg Bridge Problem

Is it possible to find a route through the graph that starts and ends at some vertex and traverses each edge exactly once?

In other words, does the graph have an Euler circuit?

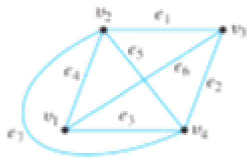


NO

8

Example 4

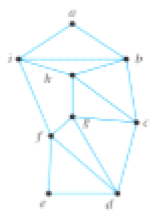
Show that the graph below does not have an Euler circuit.



v_3 has degree 3

Example 5

Verify that the graph has an Euler circuit. Then, describe two distinct Euler circuits starting and ending at vertex a.



a b c d e f g h i f d g c h b i a

Euler Trails

Let G be a graph and let v and w be two distinct vertices of G. An Euler trail from v to w is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

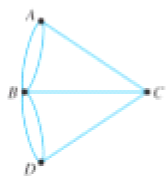
9

Theorem

Let G be a graph and let v and w be two distinct vertices of G. There is an Euler trail from v to w iff G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

The Königsberg Bridge Problem

Does the graph have an Euler trail?



NO

Hamiltonian Circuits

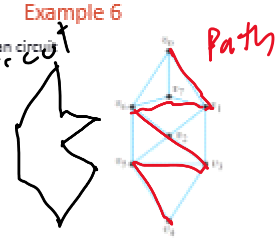
Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G.

In other words, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once except for the first and last, which are the same.

Example 6

Find a Hamiltonian circuit for the graph

circuit



Traveling salesman Problem

$V_0 \; V_1 \; V_7 \; V_6 \; V_2 \; V_3 \; V_5 \; V_4$

Hamiltonian Circuits

If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

i. H contains every vertex of G

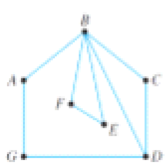
ii. H is connected

iii. H has the same number of edges and vertices

iv. Every vertex of H has degree 2.

Example 7

Show that the graph cannot have a Hamiltonian circuit.



11

Handshake Theorem and Trees

Wednesday, December 7, 2022 1:59 PM

CS1200+Lecture+32+...

Sections 4.9 and 10.4

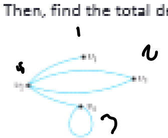
The Handshake Theorem
An Introduction to Trees

Total Degree of a Graph

The total degree of a graph is the sum of the degrees of all the vertices of the graph.

Example 1

Find the degree of each vertex of the graph G shown below. Then, find the total degree of the graph.



1+2+3+4= 10

The Handshake Theorem

If G is any graph, then the sum of the degrees of all of the vertices of G equals twice the number of edges of G.

Corollary: Total Degree of a Graph

The total degree of a graph is even.

Example 2

Either draw a graph with the specified properties or show that no such graph exists.

a) A graph with four vertices of degrees 1, 1, 2, and 3.

b) A graph with four vertices of degrees 1, 1, 3, and 3.

2

Corollary: Vertices of Odd Degree

In any graph there is an even number of vertices of odd degree.

Example 3

In a group of 15 people, is it possible for each person to have exactly 3 friends?

NO

Simple Graphs

A simple graph is a graph that does not have any loops or parallel edges.

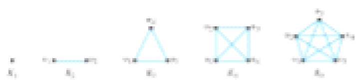
Example 4

Either draw a graph with the specified properties or explain why no such graph exists.

A simple graph with six edges and all vertices of degree 3.

Complete Graphs

Let n be a positive integer. A complete graph on n vertices, denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.



Trees

A graph is said to be circuit-free iff it has no circuits.

A graph is called a tree iff it is circuit-free and connected.

A trivial tree is a graph that consists of a single vertex.

A graph is called a forest iff it is circuit-free and not connected.

Parse Trees

In the study of grammars, trees are often used to show the derivation of grammatically correct sentences from certain basic rules. Such trees are called syntactic derivation trees or parse trees.

Example 5

A very small subset of English grammar, for example, specifies that

1. a sentence can be produced by writing first a noun phrase and then a verb phrase;
2. a noun phrase can be produced by writing an article and then a noun;
3. a noun phrase can also be produced by writing an article, then an adjective, and then a noun;
4. a verb phrase can be produced by writing a verb and then a noun phrase.

Example 5

The derivation of the sentence "The young man caught the ball" from the above rules is described by the tree shown below.

Trees in Linguistics

In the study of linguistics, syntax refers to the grammatical structure of sentences, and semantics refers to the meanings of words and their interrelations.

A sentence can be syntactically correct but semantically incorrect, as in the nonsensical sentence "The young ball caught the man," which can be derived from the rules given on the previous slides.

A sentence can contain syntactic errors but not semantic ones, as, for instance, when a two-year-old child says, "Me hungry!"

Lemma

Any tree that has more than one vertex has at least one vertex of degree 1.

Characterizing Trees

Let T be a tree.

A vertex of degree 1 in T is called a leaf (or a terminal vertex).

A vertex of degree greater than 1 in T is called an internal vertex (or a branch vertex).

Theorems

For any positive integer n, any tree with n vertices has $n - 1$ edges.

For any positive integer n, if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Corollary

If G is any graph with n vertices and m edges (where m and n are positive integers) and $m \geq n$, then G has a circuit.

Example 6

A connected graph has twelve vertices and eleven edges. Does it have a vertex of degree 1?

Yes